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PERIODIC SOLUTIONS FOR A SYSTEM OF FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH PRODUCT OF TWO NONLINEAR FUNCTIONS AND WITH A FUNCTION INVOLVING VARIABLE DEVIATIONS UNDER THE MAXIMUM

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Abstract. This paper investigates periodic solutions of a nonlinear system of fractional functional integro-differential equations involving the Gerasimov--Caputo fractional operator, a product of two nonlinear functions, and maxima-type nonlinearities with variable deviations. The considered model combines hereditary effects, nonlocal interactions, and extremal functional dependencies, which substantially complicate the qualitative analysis of the system. The original periodic boundary-value problem is transformed into an equivalent system of nonlinear functional-integral equations. Explicit estimates for the associated nonlinear operators are obtained, and a constructive iterative procedure based on the method of successive approximations is developed. Sufficient conditions guaranteeing existence and uniqueness of periodic solutions are established by combining the contraction mapping principle with detailed operator estimates in an appropriate Banach space. Furthermore, the periodic solvability problem is reduced to the investigation of zeros of an associated nonlinear vector field. Using homotopy arguments and topological index techniques, sufficient conditions for the existence of periodic solutions are derived. The obtained results provide a constructive framework for the analysis of nonlinear fractional systems with memory, delays, maxima, and nonlocal interactions.

Key words: Periodic solutions, system of functional integro-differential equations, product of two nonlinear functions, function involving variable deviations under the maximum.

ПЕРИОДИЧЕСКИЕ РЕШЕНИЯ СИСТЕМЫ ФУНКЦИОНАЛЬНЫХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ПРОИЗВЕДЕНИЕМ ДВУХ НЕЛИНЕЙНЫХ ФУНКЦИЙ И ФУНКЦИЕЙ С ПЕРЕМЕННЫМИ ОТКЛОНЕНИЯМИ ПОД МАКСИМУМАМИ

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Аннотация. В данной работе исследуются периодические решения нелинейной системы дробных функциональных интегро-дифференциальных уравнений, включающей дробный оператор Герасимова—Капуто, произведение двух нелинейных функций и нелинейности типа максимумов с переменными отклонениями. Рассматриваемая модель сочетает в себе наследственные эффекты, нелокальные взаимодействия и экстремальные функциональные зависимости, что существенно усложняет качественный анализ системы. Исходная периодическая краевая задача преобразуется в эквивалентную систему нелинейных функционально-интегральных уравнений. Получены явные оценки для соответствующих нелинейных операторов и разработана конструктивная итерационная процедура, основанная на методе последовательных приближений. Достаточные условия, гарантирующие существование и единственность периодических решений, установлены путем объединения принципа сжимающих отображений с подробными оценками операторов в соответствующем банаховом пространстве. Кроме того, периодическая задача разрешимости сводится к исследованию нулей соответствующего нелинейного векторного поля. С помощью гомотопических аргументов и топологических индексных методов выведены достаточные условия существования периодических решений. Полученные результаты обеспечивают конструктивную основу для анализа нелинейных дробных систем с памятью, задержками, максимумами и нелокальными взаимодействиями.

Ключевые слова: Периодические решения, система функциональных интегро-дифференциальных уравнений, произведение двух нелинейных функций, функция, содержащая переменные отклонения под максимумом.

Introduction

Differential and integro-differential equations have become one of the principal mathematical tools for describing dynamical systems with memory, hereditary properties, anomalous transport phenomena, and nonlocal interactions. During the last decades, such equations have found numerous applications in viscoelasticity, control theory, continuum mechanics, biology, signal processing, heat transfer, and many other branches of modern science and engineering. In addition, in the theory of differential equations, one of the main directions is functional-differential equations (Artykova et al., 2026; Gladilina and Ignatyev, 2004; Mamanov and et al., 2026).

A particularly important role in the qualitative theory of dynamical systems is played by periodic solutions. Periodic regimes describe recurrent processes, oscillatory behavior, and stable long-term patterns that arise naturally in both natural and engineered systems. Consequently, the study of periodic solutions remains one of the central directions in the theory of differential and integro-differential equations, see, for example: (Bai and Yang, 2007; Chen and et al., 2007; Hu and Han, 2009; Li and et al., 2015; Ma and et al., 2012).

In the theory of fractional order differential equations, the main place is given not only to theoretical development, but also to practical application in various fields of science and practice, see, for example: (Area and et al., 2015; Delbosco and Rodino, 1996; Handbook of Fractional Calculus with Applications 2019; Hilfer, 2000; Hussain and et al., 2020; Kaufmann and Mboumi, 2007; Kilbas and et al., 2006; Kosmakova and et all., 2026; Samko and et al., 1993; Sun and et al., 2019). At the same time, many practical models involve not only memory effects represented by fractional operators but also functional dependencies containing maxima of the unknown function over variable intervals. Such maxima-type nonlinearities appear in control systems with threshold mechanisms, biological models with delayed responses, optimization problems, and engineering systems with extremal feedback. The simultaneous presence of fractional memory effects and maxima considerably increases the analytical complexity of the problem.

An additional difficulty arises when nonlinear integral interactions are represented by products of nonlinear functions. In this case, the resulting equations combine several distinct sources of nonlinearity: fractional memory, integral dependence, variable deviations, and extremal operators. The interaction of these mechanisms leads to mathematical models whose qualitative properties remain insufficiently understood.

Motivated by these considerations, the present paper studies a periodic boundary-value problem for a nonlinear system of fractional functional integro-differential equations involving the Gerasimov-Caputo operator, maxima-type nonlinearities, and a product of two nonlinear functions. The considered model includes various types of deviating arguments and therefore unifies several important classes of functional differential equations. Related problems of finding periodic solutions to integer-order nonlinear functional differential equations are considered in (Yuldashev, 2022a; Yuldashev, 2022b; Yuldashev and Abduvahobov, 2023; Yuldashev and Sulaimonov, 2022).

The main idea of the paper consists in reducing the original periodic boundary-value problem to an equivalent system of nonlinear functional-integral equations. This reduction makes it possible to construct a convergent iterative process and to apply fixed-point techniques. Existence and uniqueness of periodic solutions are established by means of the contraction mapping principle. Furthermore, the periodic solvability problem is connected with the investigation of zeros of an associated nonlinear vector field. By combining homotopy arguments with topological index methods, additional sufficient conditions for the existence of periodic solutions are obtained.

The developed approach provides a constructive analytical framework for studying nonlinear periodic processes in fractional systems with memory, maxima, and nonlocal interactions.

1. Problem statement

In this paper on the interval $[0,T]$ we consider the questions of existence and constructive methods of calculating of the periodic solutions of the system of nonlinear ordinary first order integro-differential equations with maxima

$$\begin{aligned} {}_C D_{0t}^{\alpha} x(t) &= q(t, x(t)) + \\ &+\int_0^t g(s, x(s))\, f\left(s, x(s), \int_{-\infty}^{s} K\left(s, \theta, \max\left\{ x(\tau) \middle| \tau \in P(\theta) \right\} \right) d\theta \right) ds, \end{aligned} \tag{1}$$

where

$${}_C D_{0t}^{\alpha} x(t) = I_{0t}^{1-\alpha} x'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(s)\, ds}{(t-s)^{\alpha}}, \ \ t \in (0,T)$$

is the Gerasimov-Caputo fractional differential operator,

$$I_{0t}^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds, \quad 0<\alpha\le 1$$

is Riemann-Liouville integral operator, α is order of fractional differential operator, $\Gamma(\alpha)$ is Gamma function.

$x\in X$, X is the closed bounded domain in the space R^n, ∂X is its border, $g\in C([0,T]\times X, R^n)$ $f\in C([0,T]\times X\times R^n, R^n)$, functions q,g,f are T-periodic, $P(t)\equiv[\lambda_1(t),\lambda_2(t)]$, $\lambda_i(t)\in C[0,T]$, $i=1,2.$

According to the deviation functions are possible following cases:

1). $-\infty<\lambda_1(t)<\lambda_2(t)\le 0$ (initial argument case);

2). $-\infty<\lambda_1(t)<\lambda_2(t)\le t$ (initial-delay argument case);

3). $-\infty<\lambda_1(t)<0,\ \ 0<\lambda_2(t)\le t$ (initial-delay argument case);

4). $-\infty<\lambda_1(t)\le t<\lambda_2(t)\le T$ (initial-delay-advancing argument case);

5). $-\infty<\lambda_1(t)\le t<\lambda_2(t)<\infty$ (initial-delay-advancing-final argument case);

6). $0\le\lambda_1(t)<\lambda_2(t)\le t$ (delay argument case);

7). $0<\lambda_1(t)\le t<\lambda_2(t)\le T$ (delay-advancing argument case);

8). $t\le\lambda_1(t)<\lambda_2(t)\le T$ (advancing argument case);

9). $0<\lambda_1(t)\le T<\lambda_2(t)<\infty$ (advancing-final argument case);

10). $-\infty<\lambda_1(t)\le T<\lambda_2(t)<\infty$ (initial-advancing-final argument case);

11). $t\le\lambda_1(t)<\lambda_2(t)<\infty$ (advancing-final argument case).

We consider the case 5. Other cases will be study similarly. The choice of the initial conditions for equation (1) depends from the nature of the deviations. In the case of 5 we will study the equation (1) with the following conditions

$$\begin{cases} x(t)=\varphi_1(t), & t\in(-\infty,0],\\ x(t)=\varphi_2(t), & t\in[T,\infty),\\ x(0)=x(T), \end{cases} \tag{2}$$

where $\varphi_1(t)\in C\Big((-\infty,0],R^n\Big)$, $\varphi_2(t)\in C\Big([T,\infty),R^n\Big)$, $\int_{-\infty}^{t}\big|K(t,s,x)\big|ds<\infty.$

By $C\Big([0,T],R^n\Big)$ denoted the Banach space, which consists of continuous vector functions $x(t)$, defined on the segment $[0,T]$, with values in R^n and with the norm

$$\|x\|_{C[0,T]}=\sqrt{\sum_{j=1}^{n}\max_{0\le t\le T}\big|x_j(t)\big|^2}.$$

By the $BD\Big([0,T],R^n\Big)$ we denote the Banach space of continuous functions on the interval $[0,T]$ with the norm

$$\|x(t)\|_{BD[0,T]}=\|x(t)\|_{C[0,T]}+h\cdot\|x'(t)\|_{C[0,T]},\quad 0<h=const.$$

Formulation of problem. To find the T-periodic function $x(t)\in BD\left([0,T],R^n\right)$, which for all $t\in[0,T]$ satisfies the system of integro-differential equations (1) and the periodic condition in (2) and goes through x_0 at $t=0$.

2. Reduction to functional-integral equation

Let the function $x(t)\in BD\left([0,T],R^n\right)$ is a solution of the periodic boundary value problem (1), (2). Using the definition of the operator, we integrate of the equation (1) on the given interval $(0,t)$

$$I_{0t}^{\alpha}{}_C D_{0t}^{\alpha}\left(x(t)\right)=I_{0t}^{\alpha}I_{0t}^{1-\alpha}x'(t)=I_{0t}^{1}x'(t)=\int_0^t x'(s)ds=x(t)-x(0), \tag{3}$$

$$I_{0t}^{\alpha}\eta\left(t,x(t)\right)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta\left(s,x(s)\right)ds. \tag{4}$$

Appling the formulas (3) and (4) to the last fractional integro-differential equation, we obtain

$$x(t)=x(0)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[q(s,x)+\int_0^s g(\theta,x)f(\theta,x,y)d\theta\right]ds, \tag{5}$$

where

$$g(t,x)=g(t,x(t)),\quad q(t,x)=q(t,x(t)),\quad f(t,x,y)=f\left(t,x(t),\int_{-\infty}^t K\left(t,s,\max\left\{x(\tau)\middle|\tau\in P(s)\right\}\right)ds\right).$$

The quantity $x(0)$ in the equation (5) is unknown. To find it we use the periodic condition (2). So, from (5) we have

$$x(T)=x(0)+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left[q(s,x)+\int_0^s g(\theta,x)f(\theta,x,y)d\theta\right]ds.$$

Hence, taking the condition (2) into account, we obtain:

$$\int_0^T (T-s)^{\alpha-1}\left[q(s,x)+\int_0^s g(\theta,x)f(\theta,x,y)d\theta\right]ds=0.$$

Consequently, the differential equation (1) one can write as

$$\begin{gathered}
{}_C D_{0t}^{\alpha}x(t)=q(t,x(t))+\\
+\int_0^t g(s,x(s))f\left(s,x(s),\int_{-\infty}^s K\left(s,\theta,\max\left\{x(\tau)\middle|\tau\in P(\theta)\right\}\right)d\theta\right)ds-\\
-\frac{1}{T}\int_0^T (T-s)^{\alpha-1}q(s,x(s))ds-\\
-\frac{1}{T}\int_0^T (T-s)^{\alpha-1}\int_0^s g(\theta,x(\theta))f\left(\theta,x(\theta),\int_{-\infty}^{\theta} K\left(\theta,\varsigma,\max\left\{x(\tau)\middle|\tau\in P(\varsigma)\right\}\right)d\varsigma\right)d\theta ds.
\end{gathered} \tag{6}$$

Then by integration of the equation (6), we obtain the following system of equations:

$$x(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[q(s,x(s))+$$

$$+\int_0^s g(\theta,x(\theta))\, f\left(\theta,x(\theta),\int_{-\infty}^{\theta} K\left(\theta,\varsigma,\max\left\{x(\tau)\middle|\tau\in P(\varsigma)\right\}\right)d\varsigma\right)d\theta-$$

$$-\frac{1}{T}\int_0^T (T-\theta)^{\alpha-1} q(\theta,x(\theta))d\theta-\frac{1}{T}\int_0^T (T-\theta)^{\alpha-1}\int_0^{\theta} g(\varsigma,x(\varsigma))\times$$

$$\times f\left(\varsigma,x(\varsigma),\int_{-\infty}^{\varsigma} K\left(\varsigma,\xi,\max\left\{x(\tau)\middle|\tau\in P(\xi)\right\}\right)d\xi\right)d\varsigma d\theta\Big]ds. \tag{7}$$

Lemma 1. For the equation (7) is true the following estimate

$$\| x(t)-x_0 \|_{C[0,T]} \le T^{\alpha}\frac{T+(1+\alpha)}{\alpha(1+\alpha)\Gamma(\alpha)}\left(M_q+TM_g M_f\right), \tag{8}$$

where

$$M_q=\| q(t,x(t))\|_{C[0,T]},\quad M_g=\| g(t,x(t))\|_{C[0,T]},\quad M_f=\| f(t,x(t),y(t))\|_{C[0,T]}.$$

Proof. We rewrite the equation (7) as

$$x(t)-x_0=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} q(s,x(s))ds+$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s g(\theta,x(\theta))\, f\left(\theta,x(\theta),\int_{-\infty}^{\theta} K\left(\theta,\varsigma,\max\left\{x(\tau)\middle|\tau\in P(\varsigma)\right\}\right)d\varsigma\right)d\theta ds-$$

$$-\frac{1}{T}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\int_0^T (T-\theta)^{\alpha-1} q(\theta,x(\theta))d\theta-\frac{1}{T}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\int_0^T (T-\theta)^{\alpha-1}\times$$

$$\times\int_0^{\theta} g(\varsigma,x(\varsigma))\, f\left(\varsigma,x(\varsigma),\int_{-\infty}^{\varsigma} K\left(\varsigma,\xi,\max\left\{x(\tau)\middle|\tau\in P(\xi)\right\}\right)d\xi\right)d\varsigma d\theta.$$

Hence, implies that there is true the following estimate

$$\| x(t)-x_0 \|_{C[0,T]} \le \alpha(t)\Big[\| q(t,x(t))\|+T\| g(t,x(t))\|\| f(t,x(t),y(t))\|\Big], \tag{9}$$

where

$$\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(1+s)ds-\frac{1}{\Gamma(\alpha)}\frac{1}{T}\int_0^t (t-s)^{\alpha-1}ds\int_0^T (T-\theta)^{\alpha-1}(1+\theta)d\theta.$$

We calculate the integrals:

$$A_1=\int_0^t (t-s)^{\alpha-1}sds=\left|\begin{array}{l} s=tz\\ s=0\Rightarrow z=0\\ s=t\Rightarrow z=1\end{array}\right|=\int_0^1 t^{\alpha-1}(1-z)^{\alpha-1}tztdz=t^{\alpha+1}B(2,\alpha),$$

$$A_2 = \int_0^t (t-s)^{\alpha-1} ds = \left| \begin{array}{l} t-s = z \Rightarrow ds = -dz \\ s = 0 \Rightarrow z = t \\ s = t \Rightarrow z = 0 \end{array} \right| = -\int_t^0 z^{\alpha-1} dz = \frac{t^\alpha}{\alpha}.$$

Similarly, we obtain that

$$A_3 = \int_0^T (T-s)^{\alpha-1} s ds = T^{\alpha+1} B(2,\alpha), \quad A_4 = \int_0^T (T-s)^{\alpha-1} ds = \frac{T^\alpha}{\alpha},$$

$$A_5 = \int_0^t (t-s)^{\alpha-1} \int_0^T (T-\theta)^{\alpha-1} (1+\theta) d\theta ds = A_2(A_3 + A_4) = \frac{(tT)^\alpha}{\alpha} \left(T \frac{\Gamma(\alpha)}{\Gamma(2+\alpha)} + \frac{1}{\alpha} \right).$$

Then we have the following estimate

$$\alpha(t) = \frac{1}{\Gamma(\alpha)} [A_1 + A_2] - \frac{1}{\Gamma(\alpha)} \frac{1}{T} A_5 = \frac{t^\alpha}{\Gamma(\alpha)} \left[t \frac{\Gamma(\alpha)}{\Gamma(2+\alpha)} + \frac{1}{\alpha} \right] -$$

$$- \frac{1}{\Gamma(\alpha)} \frac{t^\alpha T^{\alpha-1}}{\alpha} \left(T \frac{\Gamma(\alpha)}{\Gamma(2+\alpha)} + \frac{1}{\alpha} \right) \le T^\alpha \frac{T + (1+\alpha)}{\alpha(1+\alpha)\Gamma(\alpha)}.$$

It is easy to check that from (9) follows (8). Lemma 1 is proved.

Remark. T -periodic solution $x_{\varphi_1(t)} = \psi(t)$ of the system (1) with initial value function $\varphi_1(t)$ on the initial set $(-\infty, 0]$ is defined by the initial value function $\varphi_1(t)$, which is periodical continuation of the solution $\psi(t)$ into initial set $(-\infty, 0]$. T -periodic solution $x_{\varphi_2(t)} = \psi(t)$ of the system (1) with final value function $\varphi_2(t)$ on the final set $[T, \infty)$ is defined by the final value function $\varphi_2(t)$, which is periodical continuation of the solution $\psi(t)$ into final set $[T, \infty)$.

Lemma 2. For the difference of two functions with maxima there holds the following estimate

$$\left\| \max\left\{ x(\tau) \middle| \tau \in P(t) \right\} - \max\left\{ y(\tau) \middle| \tau \in P(t) \right\} \right\|_{C[0,T]} \le$$

$$\le \| x(t) - y(t) \|_{C[0,T]} + h \left\| \frac{\partial}{\partial t} [x(t) - y(t)] \right\|_{C[0,T]},$$

where $P(t) \equiv [\lambda_1(t), \lambda_2(t)]$,

$$h = \max\left\{ \sup_{-\infty<t\le T} |\lambda_1(t)| + \sup_{0\le t<\infty} |\lambda_2(t)|; \ \sup_{-\infty<t\le 0} |\varphi_1(t)| + \sup_{T\le t<\infty} |\varphi_2(t)| \right\}.$$

3. Main results

Theorem 1. Assume that for all $t \in [0,T]$ are fulfilled the following conditions:

1. $\| q(t, x(t)) \| \le M_q < \infty, \ \| g(t, x(t)) \| \le M_g < \infty, \ \| f(t, x(t), y(t)) \| \le M_f < \infty;$
2. $\| q(t, x_1) - q(t, x_2) \| \le L_1 \| x_1 - x_2 \|;$
3. $\| g(t, x_1) - g(t, x_2) \| \le L_2 \| x_1 - x_2 \|;$
4. $\| f(t, x_1, y_1) - f(t, x_2, y_2) \| \le L_3 [\| x_1 - x_2 \| + \| y_1 - y_2 \|];$
5. $\| K(t, s, x_1) - K(t, s, x_2) \| \le L_4(s) \| x_1 - x_2 \|, \ 0 < \sup_t \int_{-\infty}^t L_4(s) ds < \infty;$

6. The radius of the inscribed ball in X is greater than $T^{\alpha}\dfrac{T+(1+\alpha)}{\alpha(1+\alpha)\Gamma(\alpha)}\left(M_q+TM_gM_f\right)$;

7. $\rho=\chi_0\chi_1<1,$ where $\chi_0=\max\left\{L_1+T(M_gL_3+M_fL_2);M_0M_gL_3\right\},$

$$\chi_1=\max\left\{T^{\alpha}\frac{T+(1+\alpha)}{\alpha(1+\alpha)\Gamma(\alpha)};\alpha_0'\right\},\; M_0=\sup_t\int_{-\infty}^{t}L_4(s)ds<\infty.$$

If the system (1) has a solution for all $t\in[0,T]$, then this solution can be founded by the system of nonlinear functional-integral equations

$$x(t,x_0)=J(t;x)\equiv x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}q(s,x(s,x_0))ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\int_0^s g\left(\theta,x(\theta,x_0)\right)\times$$
$$\times f\left(\theta,x(\theta,x_0),\int_{-\infty}^{\theta}K\left(\theta,\varsigma,\max\left\{x(\tau,x_0)\middle|\tau\in P(\varsigma)\right\}\right)d\varsigma\right)d\theta ds-$$
$$-\frac{1}{T}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\int_0^T(T-\theta)^{\alpha-1}q(\theta,x(\theta,x_0))d\theta-\frac{1}{T}\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\int_0^T(T-\theta)^{\alpha-1}\times$$
$$\times\int_0^{\theta}g\left(\varsigma,x(\varsigma,x_0)\right)f\left(\varsigma,x(\varsigma,x_0),\int_{-\infty}^{\varsigma}K\left(\varsigma,\xi,\max\left\{x(\tau,x_0)\middle|\tau\in P(\xi)\right\}\right)d\xi\right)d\varsigma d\theta. \qquad (10)$$

Proof. The theorem we proof by the method of successive approximations, defining iteration process as

$$x_0(t,x_0)=x_0,\; x_{m+1}(t,x_0)=J(t;x_m),\quad m=0,1,2,3,... \qquad (11)$$

We will show that the right-hand side of the system of equations (10) as an operator maps a ball with radius $T^{\alpha}\dfrac{T+(1+\alpha)}{\alpha(1+\alpha)\Gamma(\alpha)}\left(M_q+TM_gM_f\right)$ into itself and is a contraction operator. So, according to the lemma 1, from (11) we have

$$\| x_{m+1}(t,x_0)-x_0\|_{C[0,T]}\le T^{\alpha}\frac{T+(1+\alpha)}{\alpha(1+\alpha)\Gamma(\alpha)}\left(M_q+TM_gM_f\right). \qquad (12)$$

By integration of the system of integral equations (10) we obtain

$$x'_{m+1}(t,x_0)=\frac{1}{\Gamma(\alpha)}\int_0^t(1-s)^{\alpha-1}q(s,x_m(s,x_0))ds+\frac{1}{\Gamma(\alpha)}\int_0^t(1-s)^{\alpha-1}\int_0^s g\left(\theta,x_m(\theta,x_0)\right)\times$$
$$\times f\left(\theta,x_m(\theta,x_0),\int_{-\infty}^{\theta}K\left(\theta,\varsigma,\max\left\{x_m(\tau,x_0)\middle|\tau\in P(\varsigma)\right\}\right)d\varsigma\right)d\theta ds-$$
$$-\frac{1}{T}\frac{1}{\Gamma(\alpha)}\int_0^t(1-s)^{\alpha-1}ds\int_0^T(T-\theta)^{\alpha-1}q(\theta,x_m(\theta,x_0))d\theta-$$
$$-\frac{1}{T}\frac{1}{\Gamma(\alpha)}\int_0^t(1-s)^{\alpha-1}ds\int_0^T(T-\theta)^{\alpha-1}\int_0^{\theta}g\left(\varsigma,x_m(\varsigma,x_0)\right)\times$$

$$\times f\left(\varsigma, x_m(\varsigma, x_0), \int\limits_{-\infty}^{\varsigma} K\left(\varsigma, \xi, \max\left\{x_m(\tau, x_0) \middle| \tau \in P(\xi)\right\}\right) d\xi\right) d\varsigma d\theta. \tag{13}$$

From (13) we get

$$\| x'_{m+1}(t, x_0) \|_{C[0,T]} \le \alpha'(t)\left(M_q + TM_g M_f\right), \tag{14}$$

where

$$\alpha'(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (1-s)^{\alpha-1}(1+s)ds - \frac{1}{\Gamma(\alpha)} \frac{1}{T} \int\limits_0^t (1-s)^{\alpha-1} ds \int\limits_0^T (T-\theta)^{\alpha-1}(1+\theta)d\theta.$$

We calculate the integrals:

$$B_1 = \int\limits_0^t (1-s)^{\alpha-1} ds = -\int\limits_0^t (1-s)^{\alpha-1} d(1-s) = \frac{1-(1-t)^{\alpha}}{\alpha},$$

$$B_2 = \int\limits_0^t (1-s)^{\alpha-1} s ds = -\int\limits_0^t (1-s)^{\alpha-1}[(1-s)-1]ds =$$

$$= -\int\limits_0^t (1-s)^{\alpha} ds + \int\limits_0^t (1-s)^{\alpha-1} ds = -\frac{1-(1-t)^{\alpha+1}}{\alpha+1} + \frac{1-(1-t)^{\alpha}}{\alpha} =$$

$$= \frac{1-(1+\alpha)(1-t)^{\alpha} + \alpha(1-t)^{\alpha+1}}{\alpha(1+\alpha)}.$$

Similarly, we obtain that

$$B_3 = \int\limits_0^T (T-s)^{\alpha-1} s ds = T^{\alpha+1} B(2,\alpha) = T^{\alpha+1} \frac{\Gamma(\alpha)}{\Gamma(2+\alpha)}, \quad B_4 = \int\limits_0^T (T-s)^{\alpha-1} ds = \frac{T^{\alpha}}{\alpha},$$

$$B_5 = \int\limits_0^t (1-s)^{\alpha-1} ds \int\limits_0^T (T-\theta)^{\alpha-1}(1+\theta)d\theta = B_1(B_3 + B_4) =$$

$$= T^{\alpha} \frac{1-(1-t)^{\alpha}}{\alpha}\left(T \frac{\Gamma(\alpha)}{\Gamma(2+\alpha)} + \frac{1}{\alpha}\right).$$

Then we have the following estimate

$$\alpha'(t) = \frac{1}{\Gamma(\alpha)}\left[B_1 + B_2\right] - \frac{1}{\Gamma(\alpha)} \frac{1}{T} B_5 = \frac{1}{\Gamma(\alpha)}\left[\frac{1-(1-t)^{\alpha}}{\alpha} + \frac{1-(1+\alpha)(1-t)^{\alpha} + \alpha(1-t)^{\alpha+1}}{\alpha(1+\alpha)}\right] -$$

$$-\frac{T^{\alpha-1}}{\Gamma(\alpha)} \frac{1-(1-t)^{\alpha}}{\alpha}\left(T \frac{\Gamma(\alpha)}{\Gamma(2+\alpha)} + \frac{1}{\alpha}\right) \le \alpha'_0,$$

where

$$\alpha'_0 = \begin{cases} \dfrac{\alpha(1-t)^{\alpha+1}}{\alpha(1+\alpha)\Gamma(\alpha)}, & T < 1, \\ \dfrac{2+\alpha}{\alpha(1+\alpha)\Gamma(\alpha)}, & T = 1, \\ \dfrac{1-(1-t)^{\alpha}}{\alpha\Gamma(\alpha)} + \dfrac{1-(1+\alpha)(1-t)^{\alpha}}{\alpha(1+\alpha)}, & T > 1. \end{cases}$$

We consider a difference $x_{m+1}(t,x_0)-x_m(t,x_0)$, where the functions $x_{m+1}(t,x_0)$ and $x_m(t,x_0)$ are defined from the approximations of system of equations (11). By the conditions of the theorem and lemma 2, from (11) we have

$$\| x_{m+1}(t,x_0)-x_m(t,x_0)\|_{C[0,T]} \le$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big\{\Big(L_1+T(M_gL_3+M_fL_2)\Big)\| x_m(s,x_0)-x_{m-1}(s,x_0)\|+$$

$$+M_gL_3\int_{-\infty}^{s} L_4(\theta)\Big\|\max\big\{x_m(\tau,x_0)\big|\tau\in P(\theta)\big\}-\max\big\{x_{m-1}(\tau,x_0)\big|\tau\in P(\theta)\big\}\Big\|d\theta+$$

$$+\frac{1}{T}\int_0^T\Big(L_1+T(M_gL_3+M_fL_2)\Big)\Big[\| x_m(\theta,x_0)-x_{m-1}(\theta,x_0)\|+M_gL_3\int_{-\infty}^{\theta} L_4(\xi)\times$$

$$\times\Big\|\max\big\{x_m(\tau,x_0)\big|\tau\in P(\xi)\big\}-\max\big\{x_{m-1}(\tau,x_0)\big|\tau\in P(\xi)\big\}\Big\|d\xi\Big]d\theta\Big\}ds\le$$

$$\le\alpha(t)\Big[\Big(L_1+T(M_gL_3+M_fL_2)\Big)\| x_m(t,x_0)-x_{m-1}(t,x_0)\|_{C[0,T]}+$$

$$+M_0M_gL_3h\,\| x'_m(t,x_0)-x'_{m-1}(t,x_0)\|_{C[0,T]}\Big]\le T^{\alpha}\,\frac{T+(1+\alpha)}{\alpha(1+\alpha)\Gamma(\alpha)}\times$$

$$\times\chi_0\Big[\| x_m(t,x_0)-x_{m-1}(t,x_0)\|_{C[0,T]}+h\,\| x'_m(t,x_0)-x'_{m-1}(t,x_0)\|_{C[0,T]}\Big], \tag{15}$$

where

$$\chi_0=\max\big\{L_1+T(M_gL_3+M_fL_2);M_0M_gL_3\big\},\quad M_0=\sup_t\int_{-\infty}^{t} L_4(s)ds<\infty.$$

Similarly, by the assumptions of the theorem 1, from (13) we have

$$\| x'_{m+1}(t,x_0)-x'_m(t,x_0)\|_{C[0,T]}\le$$

$$\le\alpha'_0\chi_0\Big[\| x_m(t,x_0)-x_{m-1}(t,x_0)\|_{C[0,T]}+h\,\| x'_m(t,x_0)-x'_{m-1}(t,x_0)\|_{C[0,T]}\Big]. \tag{16}$$

Denoting $\chi_1=\max\left\{T^{\alpha}\,\dfrac{T+(1+\alpha)}{\alpha(1+\alpha)\Gamma(\alpha)};\alpha'_0\right\}$, from (15) and (16), we obtain

$$\| x_{m+1}(t,x_0)-x_m(t,x_0)\|_{BD[0,T]}\le\rho\cdot\| x_m(t,x_0)-x_{m-1}(t,x_0)\|_{BD[0,T]}, \tag{17}$$

where $\rho=\chi_0\chi_1$.

According to the last condition of the theorem 1, $\rho<1$. Since

$$\| x_{m+1}(t,x_0)-x_m(t,x_0)\|_{BD[0,T]}<\| x_m(t,x_0)-x_{m-1}(t,x_0)\|_{BD[0,T]},$$

from the estimate (17) we deduce that the operator on right-hand side of (11) is contracting. From the estimates (12), (14) and (15), (16) implies that there exists a unique fixed point $x(t,x_0)\in BD[0,T]$. The theorem 1 is proved.

From the estimate (17) it is easy to obtain that for $x_0,\bar{x}_0\in X$ holds

$$\| x(t,x_0)-x(t,\bar{x}_0)\|_{BD[0,T]} \le \frac{|x_0-\bar{x}_0|}{1-\rho}.$$

Now we will show the existence of periodic solutions of the system of impulsive integro-differential equations (1). We introduce designations:

$$\Delta(x_0)=\frac{1}{T}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\int_0^T (T-\theta)^{\alpha-1}q(\theta,x_\infty(\theta,x_0))d\theta+$$
$$+\frac{1}{T}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\int_0^T (T-\theta)^{\alpha-1}\int_0^\theta g\left(\varsigma,x_\infty(\varsigma,x_0)\right)\times$$
$$\times f\left(\varsigma,x_\infty(\varsigma,x_0),\int_{-\infty}^{\varsigma} K\left(\varsigma,\xi,\max\left\{x_\infty(\tau,x_0)\middle|\tau\in P(\xi)\right\}\right)d\xi\right)d\varsigma d\theta, \tag{18}$$

$$\Delta_m(x_0)=\frac{1}{T}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\int_0^T (T-\theta)^{\alpha-1}q(\theta,x_m(\theta,x_0))d\theta+$$
$$+\frac{1}{T}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\int_0^T (T-\theta)^{\alpha-1}\int_0^\theta g\left(\varsigma,x_m(\varsigma,x_0)\right)\times$$
$$\times f\left(\varsigma,x_m(\varsigma,x_0),\int_{-\infty}^{\varsigma} K\left(\varsigma,\xi,\max\left\{x_m(\tau,x_0)\middle|\tau\in P(\xi)\right\}\right)d\xi\right)d\varsigma d\theta, \tag{19}$$

where $x(t,x_0)=\lim\limits_{m\to\infty} x_m(t,x_0)=x_\infty(t,x_0)$ is solution of the nonlinear system (10). Therefore, $x_\infty(t,x_0)$ is the solution of the system of impulsive integro-differential equations (1) for $\Delta(x_0)=0$ going through x_0 at $t=0$. Consequently, the questions of existence of solution of the system of impulsive integro-differential equations (1) we reduce to the questions of existence of zeros of function $\Delta(x_0)$ and we solve this problem finding zeros of the function $\Delta_m(x_0)$.

Theorem 2. Assume that

1. All conditions of the theorem 1 are fulfilled;
2. There is a natural number m such that the function $\Delta_m(x_0)$ has an isolated singular point x_0^0 that $\Delta_m\left(x_0^0\right)=0$ and index of isolated singular point x_0^0 is nonzero;
3. There is a closed convex region $X_0\subset X$, containing a single singular point such that on the its border ∂X_0 is fulfilled estimate

$$\inf_{x\in\partial X_0}\|\Delta_m(x)\|\ge\left(M_q+TM_gM_f\right)\frac{\rho^{m+1}}{1-\rho}. \tag{20}$$

Then the system of impulsive integro-differential equations (1) has a periodic solution for all $t\in[0,T]$ that $x(0)\in X_0$.

Proof. By definition, index of isolated singular point x_0^0 of continuous mapping $\Delta_m(x_0)$ is equal to the characteristic of vector field, generated by mapping $\Delta_m(x_0)$ on the sufficiently small sphere S^n with the center in x_0^0. Since in X_0 there is no another singular point, which will be differenced from x_0^0 and X_0 is homeomorphic to the unit ball E^n, then characteristic of vector field $\Delta_m(x_0)$ on the sphere S^n is equal to the characteristic of this vector field on ∂X_0. The fields $\Delta_m(x_0)$ and $\Delta(x_0)$ are homotopic on ∂X_0. Let us consider families of everywhere continuous on ∂X vector fields

$$V(\sigma, x_0) = \Delta_m(x_0) + \sigma(\Delta(x_0) - \Delta_m(x_0)),$$

which connect the fields

$$V(0, x_0) = \Delta_m(x_0), \quad V(1, x_0) = \Delta(x_0).$$

We note that there is true the estimate

$$\| \Delta(x_0) - \Delta_m(x_0) \| \le \left(M_q + TM_g M_f \right) \frac{\rho^{m+1}}{1-\rho}. \tag{21}$$

Therefore, the vector field $V(\sigma, x_0)$ does not vanish anywhere on ∂X_0. Indeed, from (20) and (21) implies that

$$\| V(\sigma, x_0) \| \ge \| \Delta_m(x_0) \| - \| \Delta(x_0) - \Delta_m(x_0) \| > 0. \tag{22}$$

The fields $\Delta_m(x_0)$ and $\Delta(x_0)$ are homotopic on ∂X and the rotations of the fields homotopic on the compact are equal. Hence, taking into account (22), we conclude that the rotation of the field $\Delta(x_0)$ on the ∂X_0 is equal to the index of the singular point x_0 of the field $\Delta_m(x_0)$ and nonzero. Consequently, the vector field $\Delta(x_0)$ on the X_0 has a singular point x_0, for which $\Delta(x_0) = 0$. Therefore, the system of impulsive integro-differential equations (1) has a periodic solution for all $t \in [0, T]$ that $x(0) \in X_0$. In addition, we note that for $x_0, \bar{x}_0 \in X$ from (18) and (19) we have

$$\| \Delta(x_0) \|_{BD[0,T]} \le \frac{T^{2\alpha-1}}{\alpha \Gamma(\alpha)} \left(T \frac{\Gamma(\alpha)}{\Gamma(2+\alpha)} + \frac{1}{\alpha} \right) \left(M_q + TM_g M_f \right),$$

$$\| \Delta(x_0) - \Delta(\bar{x}_0) \|_{BD[0,T]} \le \frac{| x_0 - \bar{x}_0 |}{1-\rho}.$$

Theorem 2 is proved.

Conclusion

In this paper, a periodic boundary-value problem for a nonlinear system of fractional functional integro-differential equations with maxima and variable deviations has been investigated. The considered system combines several analytically challenging features simultaneously: fractional memory effects generated by the Gerasimov--Caputo operator, nonlinear integral interactions, products of nonlinear functions, and extremal functional dependencies.

By transforming the original problem into an equivalent system of nonlinear functional-integral equations, a constructive analytical framework for studying periodic solutions has been developed. Explicit estimates for the associated nonlinear operators were derived, which allowed the application of the method of successive approximations and the contraction mapping principle. As a result, sufficient conditions guaranteeing existence and uniqueness of periodic solutions were obtained.

A further contribution of the paper is the reduction of the periodic solvability problem to the investigation of zeros of a nonlinear vector field generated by the associated integral equations.

The application of homotopy methods and topological index theory made it possible to establish additional existence criteria for periodic solutions.

The obtained results extend existing solvability theories for fractional functional integro-differential equations and provide effective tools for the analysis of nonlinear systems with memory and extremal feedback mechanisms. The proposed methods can be further developed for broader classes of fractional systems, including impulsive equations, systems with nonlocal boundary conditions, distributed parameter models, and equations defined in infinite-dimensional functional spaces.

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