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ON THE PERIODIC SOLUTION OF THE KELLER-SEGEL MODEL OF CHEMOTAXIS WITH A LOGISTIC SOURCE

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Abstract. *The paper investigates a system of three parabolic equations, which is a model of the spatiotemporal state of two competing populations of species, both of which are chemotactically attracted by the same signal substance. Individuals move according to random diffusion and chemotaxis, and both populations reproduce themselves and mutually compete with each other according to the classical Lotka-Volterra kinetics. The global existence and uniqueness of the classical solutions of this system is proved by the contraction mapping principle using a priori Lp estimates and Schauder-type estimates for parabolic equations.*

Key words: *Keller–Segel model, chemotaxis, a priori estimates, global solution.*

О ПЕРИОДИЧЕСКОМ РЕШЕНИИ МОДЕЛИ ХЕМОТАКСИСА КЕЛЛЕРА-СЕГЕЛЯ С ЛОГИСТИЧЕСКОМ ИСТОЧНИКОМ

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Аннотация. *В статье исследуется система трех параболических уравнений, представляющая собой модель пространственно-временного состояния двух конкурирующих популяций видов, хемотаксически притягиваемых одним и тем же сигнальным веществом. Особи перемещаются в соответствии со случайной диффузией и хемотаксисом, и обе популяции воспроизводятся и взаимно конкурируют друг с другом согласно классической кинетике Лотка-Вольтерра. Глобальное существование и единственность классических решений этой системы доказывается принципом сжимающих отображений с использованием априорных оценок Lp и оценок типа Шаудера для параболических уравнений.*

Ключевые слова: *модель Келлера–Сегеля, хемотаксис, априорные оценки, глобальное решение .*

1. Introduction (Введение). It is known that in the mathematical modeling of the self-organization of living cells, the system of equations "Keller-Segel" is used [1],

$$u_t = \Delta u - \nabla(u \nabla v), \qquad (1)$$

$$v_t = \Delta v - v + u$$

The system describes the general behavior of a set of cells under the influence of chemotaxis. Under such conditions, the movement of each individual cell, although not entirely predictable, follows a preferred direction, namely to higher concentrations of a certain signaling chemical. If $u(x,t)$ is the cell density, and $v(x,t)$ is the chemical concentration, then the first equation in (1) reflects the interaction of non-directional diffusion motion, on the one hand, and the “chemotactic motion” controlled by ∇v, with others. The second equation expresses the assumption of the model that the signaling substance, in addition to diffusion and degradation, like most chemicals, is constantly produced by living cells. This association is known to be present in many other biologically significant situations associated with chemotaxis [2]. The striking feature of (1) is that, despite its simple mathematical structure, it turned out to be able to describe the phenomenon of spatial self-organization of cells.

Usually, consider two competing populations of biological species that are attracted to the same chemical stimulus. All individuals move according to the laws of random diffusion and chemotaxis, and both populations reproduce and mutually compete with each other according to the classical Lotka-Volterra kinetics .

In this note, we study a problem with periodic boundary conditions for a quasilinear system proposed by J. Tello and M. Winkler [3] and investigated in [4], which models the dynamics of populations of two competing species in the region $\Omega=\{(x,t)\ ;\ 0<x<L,t>0\}$, both of which are chemotactically attracted by the same signal substance

$$
\begin{aligned}
&u_t = (d_1u_x - \chi uw_x)_x + \mu_1(1-u-a_1v)u,\\
&v_t = (d_2v_x - \xi vw_x)_x + \mu_2(1-a_2u-v)v, \qquad (x,t)\in\Omega,\\
&w_t = w_{xx} - \lambda w + u + v, \qquad\qquad (2)\\
&u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), \quad x\in(0,L),\\
&U(0,t) = U(L,t),\ U_x(0,t) = U_x(L,t), \qquad t>0,
\end{aligned}
$$

where $U(x,t)=(u,v,w)$, $u(x,t)$and $v(x,t)$ are the population densities of two competing species in space-time (x,t) and $w(x,t)$ are concentration of the attracting substance, d_i, μ_i, a_i, $i=1,2$ λ are positive constants, χ and ξ are assumed to be non-negative constants. It is assumed that both species u and v direct their movements chemotactically along the concentration gradient of the chemical above the habitat. This is modeled by taking both $\chi>0$ and $\xi>0$. Biologically, χ and ξ measure the strength of chemical attraction to species u and v, respectively. Species kinetics is assumed to be of the classical Lotka-Volterra type, where μ_1, μ_2 measure the internal growth rate, u, v and a_1, a_2 interpret the strength of interspecific competition. In addition, the chemical is produced by both species at the same rate with no saturation effect and at the same time consumed by a particular enzyme in the environment at a rate of λ.

In the paper, first, some a priori L_p estimates and Schauder-type estimates are established. Next, we prove that model (1) has a unique classical global solution for any chemotactic coefficients $\chi, \xi>0$.

2. A priori estimates and global existence

Let us now turn to establishing L_p-estimates (u, v, w) under the above conditions. Global existence (2) is a consequence of several lemmas.

The no negativity of (u, v, w) follows from maximum principles [5].

Lemma 1. If $\mu_1, \mu_2>0$ and (u, v, w) is the only non-negative solution of equation (2) in $(0, T_{max})$, then there is a constant C>0 depending on $\|u_0, v_0, w_0\|_{L^1}$ and L such that

$$\|u(\cdot,t)\|_{L^1(0,L)} + \|v(\cdot,t)\|_{L^1(0,L)} + \|w(\cdot,t)\|_{L^1(0,L)} \le C, \textit{ for all } t\in(0, T_{max}). \qquad (3)$$

Proof. We integrate the u–equation of (1) over $(0, L)$ and have that

$$\frac{d}{dt}\int_0^L u(x,t)dx = \int_0^L \mu_1(1-u-a_1v)udx \le \int_0^L \mu_1(1-u)udx,$$

then it follows from the Gronwall's lemma that

$$\int_0^L u(x,\ t)dx \ \le\ e^{-\mu_1 t}\int_0^L u_0(x)dx\ +L,$$

similarly we can show

$$\int_0^L \mathrm{v}(x,\ t)dx \ \le\ e^{-\mu_2 t}\int_0^L \mathrm{v}_0(x)dx\ +L,$$

Integrating the w–equation over $(0,\ L)$, we easily see that $\|w(\cdot,\ t)\|_{L^1(0,L)}$ is uniformly bounded for all $t\in(0,\ \infty)$. This completes the proof of Lemma 1.

To obtain their L^∞–bounds, we shall see that it is sufficient to obtain the boundedness of $\|w_x(\cdot,\ t)\|_{L^\infty}$. For this purpose, we first convert the w–equation into the following abstract form

$$w(\cdot,\ t)\ =\ e^{(\Delta-1)t}w_0\ +\ \int_0^t e^{(\Delta-1)t}\ \ ((1-\lambda)w(\cdot,s)+u(\cdot,s)+\ v(\cdot,s))ds, \tag{4}$$

where $\Delta\ =\ \dfrac{d^{\ 2}}{dx^2}$. To estimate $w(x,\ t)$ in (4), we apply the well–known smoothing properties of operator $-\Delta+\ 1$ and estimates between the linear analytic semigroups generated by $\left\{e^{\ t\Delta}\right\}_{t\ge 0}$. We have for all $1\ \le\ p\ \le\ q\ \le\ \infty$, there exists a positive constant C dependent on $\mu_1,\ \mu_2$, and $\|w_0\|_{w^{1,q}(0,L)}$ such that

$$\|w(\cdot,\ t)\|_{w^{1,q}(0,L)}\le C(1+\int_0^t e^{-\nu(t-s)}(t-s)^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|w(\cdot,\ s)+u(\cdot,\ s)+v(\cdot,\ s)\|_{L^p}\ ds),\ \ (5)$$

where $t\in(0,\ T)$, $T\in(0,\ \infty]$, $\nu=\dfrac{\pi}{L}$ is the first Neumann eigenvalue of $-\Delta$.

Lemma 2. Let us assume the same conditions $(u_0,\ v_0,\ w_0)$ as in Lemma 1. For any $q\in(1,\ \infty)$, there exists a positive constant $C(q)$ such that

$$\|w(\cdot,\ t)\|_{w^{1,q}(0,L)}\ \le\ C(q),\ \forall t\in(0,\ T_{max}). \tag{6}$$

Proof. Let $p\ =\ 1$ in (5), **then** we have that

$$\|w(\cdot,\ t)\|_{w^{1,q}(0,L)}\le C(1+\int_0^t e^{-\nu(t-s)}(t-s)^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}(\|u(\cdot,\ s)\|_{L^1(0,L)}+$$

$$\|v(\cdot,\ s)\|_{L^1(0,L)}+\|w(\cdot,\ s)\|_{L^1(0,L)})ds)\quad . \tag{7}$$

On the other hand, there exists a constant $C_0\ >\ 0$ such that for any $q\in(1,\ \infty)$

$$\sup_{t\in(0,\infty)}\int_0^t e^{-\nu(t-s)}\ (t\ -\ s)^{-1+\frac{1}{2q}}\ ds\ <\ C_0,$$

then we conclude from (7) that

$$\|w(\cdot,\ t)\|_{w^{1,q}(0,L)}\le C(1+\sup_{s\in(0,t)}(\|u(\cdot,\ s)\|_{L^1(0,L)}+\|v(\cdot,\ s)\|_{L^1(0,L)}+\|w(\cdot,\ s)\|_{L^1(0,L)}))\ . \tag{8}$$

Finally, (6) is an immediate consequence of (1) and (8).

Lemma 3. If $p\in(2,\ \infty)$, then there is a constant $C(p)\ >\ 0$ such that

$$\|u(\cdot,\ t)\|_{L^p} + \|v(\cdot,\ t)\|_{L^p} \le C(p),\ \forall t \in (0,\ T_{max})\ . \tag{9}$$

Proof. We shall only show that $\|u(\cdot,\ t)\|_{L^p} \le C(p)$, since $\|v(\cdot,\ t)\|_{L^p} \le C(p)$, can be proved by the same arguments. For $p > 2$, we multiply the first equation of (1) by u^{p-1} and integrate it over $(0,\ L)$ by parts, then it follows from simple calculations that

$$\frac{1}{p}\frac{d}{dt}\int_0^L u^p = \int_0^L u^{p-1}u_t = \int_0^L u^{p-1}(d_1u_x - \chi u w_x)_x + \int_0^L \mu_1(1 - u - a_1v)u^p$$

$$\le -\frac{4d_1(p-1)}{p^2}\int_0^L\left|\left(u^{\frac{p}{2}}\right)_x\right|^2 + \chi\int_0^L (u^{p-1})_x u w_x - \frac{\mu_1}{2}\int_0^L u^{p+1} + C_1$$

$$= -\frac{4d_1(p-1)}{p^2}\int_0^L\left|\left(u^{\frac{p}{2}}\right)_x\right|^2 + \frac{2(p-1)}{p}\chi\int_0^L u^{\frac{p}{2}}(u^{\frac{p}{2}})_x w_x - \frac{\mu_1}{2}\int_0^L u^{p+1} + C_1, \tag{10}$$

Where C_1 is a positive constant that depends on p. It follows from Holder's and Young's inequality that

$$\int_0^L u^{\frac{p}{2}}(u^{\frac{p}{2}})_x w_x dx \le \left\|u^{\frac{p}{2}}\right\|_{L^{\frac{2(p+1)}{p}}} \left\|(u^{\frac{p}{2}})_x\right\|_{L^2} \|w_x\|_{L^{2(p+1)}} = \|u\|^{\frac{p}{2}}_{L^{p+1}} \left\|(u^{\frac{p}{2}})_x\right\|_{L^2} \|w_x\|_{L^{2(p+1)}}$$

$$\le \frac{2d_1}{p\chi}\left\|(u^{\frac{p}{2}})_x\right\|^2_{L^2} + C_2\|u\|^p_{L^{p+1}}, \tag{11}$$

In light of (11), we obtain from (10) that

$$\frac{1}{p}\frac{d}{dt}\int_0^L u^p dx \le C_3\|u\|^p_{L^{p+1}} - \frac{\mu_1}{2}\|u\|^{p+1}_{L^{p+1}} + C_1\ . \tag{12}$$

Denoting $y_p(t) = \int_0^L u^p(x,t)\,dx$, one can apply Holder's on (12) to obtain that

$$y'_p(t) \le -C_4 y_p^{\frac{p+1}{p}}(t) + C_5,\quad y_p(0) = \|u_0\|^p_{L^p}\ .$$

We conclude that $y_p(t) \le C(p)$ for all $t \in (0,\ \infty)$. Similarly, we can show that $\int_0^L u^p(x,t)\,dx \le C(p)$. This completes the proof of Lemma 3.

Theorem 4. Suppose $a_i,\ \mu_i,\ i = 1,\ 2,$ and λ are positive constants. Then for positive initial data $(u_0,\ v_0,\ w_0) \in H^1(0,\ L)\times H^1(0,\ L)\times H^1(0,\ L)$ and any constants $\chi,\ \xi \in R$, problem (1) has a unique bounded positive solution $(u(x,\ t),\ v(x,\ t),\ w(x,\ t))$ defined on $[0,\ L]\times[0,\ \infty)$ such that $(u(\cdot,t),\ v(\cdot,t),\ w(\cdot,t)) \in C([0,\infty),\ H^1(0,\ L)\times H^1(0,\ L)\times H^1(0,L))$ and $(u,\ v) \in C^{2+\alpha,2+\alpha,1+\alpha}_{loc}([0,\ L]\times[0,\ \infty))$ for some $\alpha \in \left(0,\ \frac{1}{4}\right)$.

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