## МАТЕМАТИКА

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# INTEGRAL REPRESENTATION FOR HYPERGEOMETRIC FUNCTION OF THE MITTAG-LEFFLER TYPE $\bar{F}_B^{(3)}$

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The Mittag-Leffler function has gained importance and popularity through its applications. When solving differential equations of fractional order and integral equations of fractional order. Also, the Mittag-Leffler function plays an important role in various fields of applied mathematics and engineering sciences, such as chemistry, biology, statistics, thermodynamics, mechanics, quantum physics, computer science, signal processing [1-6].

Consider the folloving three variable hypergeometric function

$$\bar{F}_B^{(3)}(x,y,z)=\bar{F}_B^{(3)}\left(\begin{matrix}a_1,\alpha_1;a_2\beta_1;a_3\gamma_1;b_1,\alpha_2;b_2,\beta_2;b_3,\gamma_2;\\ c,\alpha_3,\beta_3,\gamma_3;c_1,\alpha_4;c_2\beta_4;c_3,\gamma_4;\end{matrix}\middle| x,y,z\right)$$
$$=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{\alpha_1 m}(a_2)_{\beta_1 n}(a_3)_{\gamma_1 p}(b_1)_{\alpha_2 m}(b_2)_{\beta_2 n}(b_3)_{\gamma_2 p}}{(c)_{\alpha_3 m+\beta_3 n+\gamma_3 p}}\frac{x^m}{\Gamma(c_1+\alpha_4 m)}\frac{y^n}{\Gamma(c_2+\beta_4 n)}\frac{z^p}{\Gamma(c_3+\gamma_4 p)} \quad (1)$$

where $a_1,a_2,a_3,b_1,b_2,b_3,c,c_1,c_2,c_3,x,y,z\in\mathbb{C},\ \min\{\alpha_i,\beta_i,\gamma_i\}>0,\quad i=\overline{1,4}$

For a generalized hypergeometric function of the Mittag-Leffler type $\bar{F}_B^{(3)}$, the following integral representations of the Euler type take place

$$\bar{F}_B^{(3)}(x,y,z)=\frac{\Gamma(\mu)}{\Gamma(b_1)\Gamma(\mu-b_1)}$$
$$\times\int_0^1\xi^{b_1-1}(1-\xi)^{\mu-b_1-1}\bar{F}_B^{(3)}\left(\begin{matrix}a_1,\alpha_1;a_2\beta_1;a_3,\gamma_1;\mu,\alpha_2;b_2,\beta_2;b_3,\gamma_2;\\ c,\alpha_3,\beta_3,\gamma_3;c_1,\alpha_4;c_2\beta_4;c_3,\gamma_4;\end{matrix}\middle| x\xi^{\alpha_2},y,z\right)d\xi, \quad (2)$$

$\operatorname{Re}\mu>\operatorname{Re}b_1>0,$

$$\bar{F}_B^{(3)}(x,y,z)=\frac{\Gamma(\mu)}{\Gamma(b_2)\Gamma(\mu-b_2)}$$
$$\times\int_0^1\xi^{b_2-1}(1-\xi)^{\mu-b_2-1}\bar{F}_B^{(3)}\left(\begin{matrix}a_1,\alpha_1;a_2\beta_1;a_3,\gamma_1;b_1,\alpha_2;\mu,\beta_2;b_3,\gamma_2;\\ c,\alpha_3,\beta_3,\gamma_3;c_1,\alpha_4;c_2\beta_4;c_3,\gamma_4;\end{matrix}\middle| x,y\xi^{\beta_2},z\right)d\xi, \quad (3)$$

$\operatorname{Re}\mu>\operatorname{Re}b_2>0,$

$$\bar{F}_B^{(3)}(x,y,z)=\frac{\Gamma(\mu)}{\Gamma(b_3)\Gamma(\mu-b_3)}$$
$$\times\int_0^1 \xi^{b_3-1}(1-\xi)^{\mu-b_3-1}\bar{F}_B^{(3)}\left(\begin{matrix}a_1,\alpha_1;a_2\beta_1;a_3,\gamma_1;b_1,\alpha_2;b_2,\beta_2;\mu,\gamma_2;\\ c,\alpha_3,\beta_3,\gamma_3;c_1,\alpha_4;c_2\beta_4;c_3,\gamma_4;\end{matrix}\Big| x,y,z\xi^{\gamma_2}\right)d\xi, \quad (4)$$
$$\operatorname{Re}\mu>\operatorname{Re}b_3>0,$$

$$\bar{F}_B^{(3)}(x,y,z)=\frac{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(\mu_1-b_1)\Gamma(\mu_2-b_2)\Gamma(\mu_3-b_3)}$$
$$\times\int_0^1\int_0^1\int_0^1 \xi^{b_1-1}\eta^{b_2-1}\tau^{b_3-1}(1-\xi)^{\mu_1-b_1-1}(1-\eta)^{\mu_2-b_2-1}(1-\tau)^{\mu_3-b_3-1}\times$$
$$\times\bar{F}_B^{(3)}\left(\begin{matrix}a_1,\alpha_1;a_2\beta_1;a_3,\gamma_1;b_1,\alpha_2;\mu,\beta_2;b_3,\gamma_2;\\ c,\alpha_3,\beta_3,\gamma_3;c_1,\alpha_4;c_2\beta_4;c_3,\gamma_4;\end{matrix}\Big| x\xi^{\alpha_2},y\eta^{\beta_2},z\tau^{\gamma_2}\right)d\xi d\eta d\tau \quad (5)$$
$$\operatorname{Re}\mu_1>\operatorname{Re}b_1>0,\quad \operatorname{Re}\mu_2>\operatorname{Re}b_2>0,\quad \operatorname{Re}\mu_3>\operatorname{Re}b_3>0,$$

$$\bar{F}_B^{(3)}(x,y,z)=\frac{\Gamma(\mu)}{\Gamma(a_1)\Gamma(\mu-a_1)}$$
$$\times\int_0^1 \xi^{a_1-1}(1-\xi)^{\mu-a_1-1}\bar{F}_B^{(3)}\left(\begin{matrix}\mu,\alpha_1;a_2\beta_1;a_3,\gamma_1;b_1,\alpha_2;b_2,\beta_2;b_3,\gamma_2;\\ c,\alpha_3,\beta_3,\gamma_3;c_1,\alpha_4;c_2\beta_4;c_3,\gamma_4;\end{matrix}\Big| x\xi^{\alpha_1},y,z\right)d\xi, \quad (6)$$
$$\operatorname{Re}\mu>\operatorname{Re}a_1>0.$$

This can be proved by substituting the integral representations for the series representation of the function $\bar{F}_B^{(3)}$ and considering that the series is absolutely convergent, by interchanging the signs of integral and series and using Euler's Beta and Gamma functions.

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