МАТЕМАТИКА

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STABILITY OF THE TIME-DEPENDENT IDENTIFICATION PROBLEM FOR A FRACTIONAL TELEGRAPH EQUATION WITH THE CAPUTO DERIVATIVE

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Abstract:. *The telegraph equation* $(D_t^\rho)^2 u(t) + 2\alpha D_t^\rho u(t) + Au(t) = p(t)q + f(t)$ $(0 < t \le T, 0 < \rho < 1)$, *in a Hilbert space* H *is investigated. Here* A *is a self-adjoint, positive operator,* D_t *is the Caputo derivative. An inverse problem is considered in which, along with* $u(t)$, *also a time varying factor* $p(t)$ *of the source function is unknown. To solve this inverse problem, we take the additional condition* $B[u(t)] = \psi(t)$ *with an arbitrary bounded linear functional* B *.Existence and uniqueness theorem for the solution to the problem under consideration is proved. Inequalities of stability are obtained.*

Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Let $A: H \to H$ be an arbitrary unbounded positive selfadjoint operator in H.

At the same time, A is the operator let the inverse exist and be a compact operator. Suppose that A has a complete in H system of orthonormal eigenfunctions $\{v_k\}$ and a countable set of positive eigenvalues λ_k .It is convenient to assume that the eigenvalues do not decrease as their number increases, i.e.

$$0 < \lambda_1 \le \lambda_2 \cdots \to +\infty .$$

Let τ be an arbitrary real number. We introduce the power of operator A, acting in H according to the rule

$$A^\tau h = \sum_{k=1}^{\infty} \lambda_k^\tau h_k v_k .$$

Obviously, the domain of definition of this operator has the form

$$D(A^\tau) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 < \infty\}.$$

The fractional integration of order $\sigma < 0$ of the function $h(t)$ defined on $[0,\infty)$ has the form (see, [1]):

$$J_t^{\sigma} h(t) = \frac{1}{\Gamma(-\sigma)} \int_0^t \frac{h(\xi)}{(t-\xi)^{\sigma+1}} d\xi, \quad t > 0,$$

provided the right-hand side exists. Here $\Gamma(\sigma)$ is Euler's gamma function. Using this definition one can define the Caputo fractional derivative of order ρ,

$$D_t^{\rho} h(t) = J_t^{\rho-1} \frac{d}{dt} h(t).$$

Let $\rho \in (0,1)$ be a fixed number. Consider the following Cauchy problem

$$\begin{cases} (D_t^{\rho})^2 u(t) + 2\alpha D_t^{\rho} u(t) + Au(t) = p(t)q + f(t), \quad 0 < t \le T; \\ \lim\limits_{t\to 0} D_t^{\rho} u(t) = \varphi_0, \\ u(0) = \varphi_1, \end{cases} \tag{1.1}$$

where a part of the source function $p(t)$ is a scalar function, $f(t) \in C(H)$ and φ, q are known elements of H.

The purpose of this paper is not only to find the solution $u(t)$, but also to determine

the time-dependent part $p(t)$ of the source function. To solve this time-dependent source identification problem one needs an extra condition. Following the paper of A. Ashyralyev et al. see, [2] we consider the additional condition in a rather general form:

$$B[u(t)] = \psi(t), \quad 0 \le t \le T, \tag{1.2}$$

where $B: H \to R$ is a given bounded linear functional, and $\psi(t)$ is the given scalar function. We call the Cauchy problem (1.1) together with additional condition (1.2) the inverse problem.

When solving the inverse problem, we will investigate the Cauchy problem for various differential equations. In this case, by the solution of the problem we mean the classical solution, i.e. we will assume that all derivatives and functions involved in the equation are continuous with respect to the variable t. As an example, let us give the definition of the solution to the inverse problem.

Definition. A pair of functions $\{u(t), p(t)\}$ with the propertie $(D_t^{\rho})^2 u(t), Au(t) \in C((0,T]; H)$, $u(t) \in C(H)$, $p(t) \in C[0,T]$ and satisfying conditions (1.1), (1.2) is called the solution of the inverse problem.

Theorem 1. Let $\alpha > 0$, $Bq \ne 0$, $\varphi_0 \in H$, $\varphi_1 \in D(A^{\frac{1}{2}})$ and $(D_t^{\rho})^2 \psi(t) \in C[0,T]$. Further, let $\epsilon > 0$ be any fixed number and $q \in D(A^{1+\epsilon})$ and $f(t) \in C([0,T]; D(A^{\epsilon}))$. Then the inverse problem has a unique solution $\{u(t), p(t)\}$.

If we additionally require that the initial function φ_0, φ_1 belong to the domain of definition of operator A, then we can establish the following result on the stability of the solution to the inverse problem.

Theorem 2. Let assumptions of Theorem 1 be satisfied and let $\varphi_0 \in D(A^{\frac{1}{2}}), \varphi_1 \in D(A)$ Then the solution to the invest problem obeys the stability estimate

$$\| (D_t^{\rho})^2 u \|_{C(H)} + \| D_t^{\rho} u \|_{C(H)} + \| Au \|_{C(H)} + \| p \|_{C[0,T]} \le C_{\rho,q,B,\epsilon} \Big[\| \varphi_0 \|_{\frac{1}{2}} ++ \| \varphi_1 \|_1 + \| \psi \|_{C[0,T]} +|$$

$$+ \| D_t^{\rho} \psi \|_{C[0,T]} + \| (D_t^{\rho})^2 \psi \|_{C[0,T]} + \max_{0\le t\le T} \| f(t) \|_{\epsilon} \Big],$$

where $C_{\rho,q,B,\epsilon}$ is a constant, depending only on ρ, q, B and ϵ.

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