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ASYMPTOTICS OF THE SOLUTION TO A SYSTEM OF FOUR NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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***Abstract**. This paper investigates the solution of a singularly perturbed system of differential equations with an arbitrary degree of accuracy in a particularly critical case. The Cauchy initial value problem is considered for a singularly perturbed system of four nonlinear ordinary differential equations in the case of a stability change. The matrix of the nonlinear system possesses complex conjugate multiple eigenvalues. The problem is studied for a system of fourth-order nonlinear singularly perturbed equations in which the eigenvalues governing the stability change in an unbounded domain of the complex plane have zeros only at infinitely distant points. The proximity of the solutions of the perturbed and unperturbed problems is proved, and a corresponding estimate is derived.*

***Keywords:** Singularly perturbed system of differential equations, Cauchy problem, solution, eigenvalues of a matrix, asymptotic, critical case, uniform approximations.*

АСИМПТОТИКА РЕШЕНИЯ СИСТЕМЫ ЧЕТЫРЁХ НЕЛИНЕЙНЫХ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

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Аннотация. В данной работе исследуется решение сингулярно возмущённой системы дифференциальных уравнений с произвольной степенью точности в особо критическом случае. Рассматривается задача Коши для сингулярно возмущённой системы четырёх нелинейных обыкновенных дифференциальных уравнений в случае смены устойчивости. Матрица нелинейной системы обладает комплексно-сопряжёнными кратными собственными значениями. Исследуется задача для системы нелинейных сингулярно возмущённых уравнений четвёртого порядка, в которой собственные значения, определяющие смену устойчивости, обращаются в нуль только в бесконечно удалённых точках комплексной плоскости. Доказана близость решений возмущённой и невозмущённой задач, а также получена соответствующая оценка.

Ключевые слова: Сингулярно возмущённая система дифференциальных уравнений, задача Коши, решение, собственные значения матрицы, асимптотика, критический случай, равномерные приближения.

Introduction

The following problem is considered:

$$\varepsilon\dot{x}(t,\varepsilon) = D(t)x(t,\varepsilon) + \varepsilon f(t,x(t,\varepsilon)), \tag{1}$$

$$x(t_0,\varepsilon) = x^\circ(\varepsilon), \quad \|x^\circ(\varepsilon)\| = O(\varepsilon), \tag{2}$$

where

$$f(t,x(t,\varepsilon)) = colon(f_1(t,x(t,\varepsilon)), f_2(t,x(t,\varepsilon)), f_3(t,x(t,\varepsilon)), f_4(t,x(t,\varepsilon)))$$

is a known quantity, and D(t) is a given matrix of size 4×4.

Assume that the following conditions are satisfied:

1. For all $t \in H_0$, the matrix $D(t)$ has eigenvalues $\lambda_k(t), k = 1,2,3,4$, and admits a Jordan normal form $(D(t) \in \Phi(H_0))$.
2. $f(t,x) \in \Phi(\Delta(t,x))$, where $\Delta(t,x) = \{t_0 \in H_0, \|x\| < \delta - const\}$ does not depend on ε, and $\|f(t,x) - f(t,\tilde{x})\| \le M\|x - \tilde{x}\|$, where M is a positive constant independent of the point.
3. $\lambda_k(t) \in \Phi(H_0)$ and $\bar{\lambda}_1(t) = \lambda_3(t), \lambda_1(t) = \lambda_2(t), \lambda_3(t) = \lambda_4(t)$.

The functions $Re\,\lambda_1(t)$ and $u(t_1,t_2)$ in the domain H_0 satisfy the following conditions:

4. $Re\,\lambda_1 < 0$ on the strip

$$t \in H_{01} = \{t \in H_0 : t_0 \le t_1 < a_0, \quad -\infty < t_2 < +\infty\},$$

$Re\,\lambda_1(a_0 + i\cdot 0) = 0$ on the line $p = \{t \in H_0 : t_1 = a_0, \quad -\infty < t_2 < +\infty\}$,

$Re\,\lambda_1 > 0$ on the strip

$$t \in H_{02} = \{t \in H_0 : a_0 < t_1 \le T_0, \quad -\infty < t_2 < +\infty\}.$$

5. For all $t \in H_0$: $Im\,\lambda_1(t) > 0$.
6. $u(t_1,t_2) = 0$ on the lines $t_1 = t_0$ and $t_1 = T_0, -\infty < t_2 < +\infty$.

The objective is to find the asymptotic of the solution to problem (1), (2) under the above conditions.

Materials and Methods

Problem (1), (2) is reduced to an integral equation, and the method of successive approximations is applied to derive the estimates.

Starting from the second approximation, the paths of integration are chosen from the domain H_0. The integration paths consist of arcs of smooth curves or piecewise smooth curves.

To estimate the successive approximations, lemma 4 [1] is employed:

On the line P we will consider the equations

$$u(t_1,t_2) = at_1 + b \text{ where } a = \frac{C_1 - C_2}{T_2 - t_{01}},\ b = \frac{C_1T_2 - C_2t_{01}}{T_2 - t_{01}}, \tag{3}$$

The main lemma**:** If level lines $u_1(t_1,t_2) = C(C_2 \le C \le C_1)$completely cover line P and an arbitrary point (t_1,t_2) line P belongs to a single line-level $\{C\}, Jm\lambda_1(t) \ne 0$, then equations (6) in the line P defines valued continuously differentiable function $t_2 = \varphi(t_1)$ with the region of existing off $(t_{01} \le t_1 \le T_2)$ and the curve segment(K_0), defined by this function, connects points $(t_{01},0),(T_2,0)$, and at this $u_1(t_1,\varphi(t_1))$ is decreasing on $[t_{01},T_2]$.

The main lemma was proved with an original method. Another proof is available in our papers.

Let $C_1 = \alpha(\varepsilon), C_2 = 2\alpha(\varepsilon)$, then the domain covered by the curves K is denoted by $K_0 \subset H_0$.

Let $C_1 = -\delta, C_2 = -2\delta$, where $0 < \delta < 1$; then the corresponding domain is denoted by $K_0 = H_c$.

Let $C_1 = -\varepsilon^{p}, C_2 = -2\varepsilon^{p}$, where $0 < p < 1$. In this case the domain is denoted by $H_p \subset H_0$.

Let $K = \Delta \cap K$, where $\Delta = \{(t_1, t_2) : t_0 \le t_1 \le t_{01}, t_2 = 0\}$. The estimate of $x(t,\varepsilon)$ is carried out for points of the domain K. Here the integration path l depends on which set the points (t_1, t_2) belong to.

Selection of Integration Paths.

Let the curve for which $(t_1, t_2) \in K_0$ consist of $l = \cup^{3}{}_{k=1}\, l_k$, where l_1 is the line segment connecting the point $(t_0, 0)$ with the point $(t_{01}, 0)$; l_2 is the arc of a curve connecting the point $(t_{01}, 0)$ with the point $(t_1, t_2^*) = \tilde{\varphi}(t_1)$); and l_3 is the line segment connecting the point (t_1, t_2^*) with the point (t_1, t_2). Note that if $(t_1, t_2) \in K_0$, then on the curve K for any t_1 there exists a unique point $(t_1, t_2^*) = \varphi(t_1)$).

Estimates of the First Approximations.

We construct the first approximation satisfying the initial condition as follows:

$$\begin{cases} x_1^{(1)}(t,\varepsilon) = x_1^{(0)}(\varepsilon)\exp\left[\dfrac{1}{\varepsilon}\displaystyle\int_{t_0}^{t}\lambda_1(s)ds\right] + \displaystyle\int_{t_0}^{t} f_1\left(\tau, x^{(0)}(t,\varepsilon)\right)\exp\left[\dfrac{1}{\varepsilon}\displaystyle\int_{\tau}^{t}\lambda_1(s)ds\right]d\tau, \\ x_2^{(1)}(t,\varepsilon) = x_2^{(0)}(\varepsilon)\exp\left[\dfrac{1}{\varepsilon}\displaystyle\int_{t_0}^{t}\lambda_2(s)ds\right] + \displaystyle\int_{t_0}^{t}\left[f_2\left(\tau, x^{(0)}(t,\varepsilon)\right) + \dfrac{\eta}{\varepsilon}x_1^{(1)}(t,\varepsilon)\right]\exp\left[\dfrac{1}{\varepsilon}\displaystyle\int_{\tau}^{t}\lambda_2(s)ds\right]d\tau, \\ x_3^{(1)}(t,\varepsilon) = x_3^{(0)}(\varepsilon)\exp\left[\dfrac{1}{\varepsilon}\displaystyle\int_{t_0}^{t}\lambda_3(s)ds\right] + \displaystyle\int_{t_0}^{t} f_3\left(\tau, x^{(0)}(t,\varepsilon)\right)\exp\left[\dfrac{1}{\varepsilon}\displaystyle\int_{\tau}^{t}\lambda_3(s)ds\right]d\tau \\ x_4^{(1)}(t,\varepsilon) = x_4^{(0)}(\varepsilon)\exp\left[\dfrac{1}{\varepsilon}\displaystyle\int_{t_0}^{t}\lambda_4(s)ds\right] + \displaystyle\int_{t_0}^{t}\left[f_4\left(\tau, x^{(0)}(t,\varepsilon)\right) + \dfrac{\eta_1}{\varepsilon}x_3^{(1)}(t,\varepsilon)\right]\exp\left[\dfrac{1}{\varepsilon}\displaystyle\int_{\tau}^{t}\lambda_4(s)ds\right]d\tau. \end{cases}$$

For $(t_1, t_2) \in \overline{K}$ the following estimate is obtained:

$$|x^{(1)}(t,\varepsilon)| = O(\varepsilon^{1-p}),$$

where

$$x^{(1)}(t,\varepsilon) = colon(x_1^{(1)}(t,\varepsilon), x_2^{(1)}(t,\varepsilon), x_3^{(1)}(t,\varepsilon), x_4^{(1)}(t,\varepsilon)).$$

If $(t_1, t_2) \in \overline{K}$, then $l_1 = \bigcup_{k=1}^{3} l_k$, where l_1 is the line segment connecting the point $(t_0, 0)$ with the point $(t_0 + \gamma_1(\varepsilon), 0)$; l_2 is the arc of the curve $(\bar{C})$ connecting the point $(t_0 + \gamma_1(\varepsilon), 0)$ with the point $(t_1, t_2^*) = \tilde{\varphi}(t_1, \varepsilon)$); and l_3 is the line segment connecting the point (t_1, t_2^*) with the point (t_1, t_2).

For all $t \in H^{c}$ the estimate $\|x^{(1)}(t,\varepsilon)\| \le c\varepsilon$ holds, and for all $t \in H_4$ the estimate $\|x^{(1)}(t,\varepsilon)\| \le c\varepsilon^{1-p}$ holds.

Estimates of the Remaining Approximations.

The successive approximations $x^{(n)}(t,\varepsilon)$ with the initial condition are defined as follows:

$$x^{(n)}(t,\varepsilon) = E(t,t_0,\varepsilon)\, x^{\circ}(\varepsilon) + \int_{t_0}^{t} E(t,\tau,\varepsilon)\, f(\tau, x^{(n-1)}(\tau,\varepsilon))\, d\tau \ ,$$

where

$$E(t,t_0,\varepsilon) = exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} D(s)ds\right), \qquad E(t,\tau,\varepsilon) = exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} D(s)\, ds\right).$$

The scalar form of the system.

$$\begin{cases} x_1^{(n)}(t,\varepsilon) = x_1^0(\varepsilon) exp\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_1(s)ds + \int_{t_0}^{t} f_1\left(\tau, x_1^{(n-1)}(t,\varepsilon)\right)\left(exp\frac{1}{\varepsilon}\int_{\tau}^{t}\lambda_1(s)ds\right)d\tau, \\ x_2^{(n)}(t,\varepsilon) = x_2^0(\varepsilon) exp\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_2(s)ds + \int_{t_0}^{t} f_2\left(\tau, x_2^{(n-1)}(t,\varepsilon)\right)\left(exp\frac{1}{\varepsilon}\int_{\tau}^{t}\lambda_2(s)ds\right)d\tau, \\ x_3^{(n)}(t,\varepsilon) = x_3^0(\varepsilon) exp\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_3(s)ds + \int_{t_0}^{t} f_3\left(\tau, x_3^{(n-1)}(t,\varepsilon)\right)\left(exp\frac{1}{\varepsilon}\int_{\tau}^{t}\lambda_3(s)ds\right)d\tau, \\ x_4^{(n)}(t,\varepsilon) = x_4^0(\varepsilon) exp\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_4(s)ds + \int_{t_0}^{t} f_4\left(\tau, x_4^{(n-1)}(t,\varepsilon)\right)\left(exp\frac{1}{\varepsilon}\int_{\tau}^{t}\lambda_4(s)ds\right)d\tau. \end{cases}$$

Validity of the Estimates for $n = 2$.

For all $t \in H_c$

$$\|\tilde{x}^{(2)}(t,\varepsilon)\| \leq c\varepsilon^2,$$

and for all $t \in H_p$:

$$\|\tilde{x}^{(2)}(t,\varepsilon)\| \leq M(c\varepsilon^{1-p})^2.$$

The remaining approximations are estimated by applying the method of mathematical induction for all $t \in H_c$:

$$\|\tilde{x}^{(n)}(t,\varepsilon)\| \leq c\varepsilon^n,$$

and for all $t \in H_p$:

$$\|\tilde{x}^{(n)}(t,\varepsilon)\| \leq (c\varepsilon^{1-p})^n.$$

Theorem 1. Suppose that conditions 1–6 are satisfied. Then for problem (1), (2) there exists a unique solution admitting the representation

$$x(t,\varepsilon) = \sum_{n=1}^{\infty}\tilde{x}^{(n)}(t,\varepsilon) .$$

For $t \in H_c$ the estimate $\left\|\tilde{x}^{(n)}(t,\varepsilon)\right\| \leq c\varepsilon^n$ holds, and for $t \in H_p$ the estimate

$$\left\|\tilde{x}^{(n)}(t,\varepsilon)\right\| \leq (c\varepsilon^{1-p})^n,$$

holds, where $c > 0$ is a constant independent of t and ε.

Conclusion

As a result of this work, uniform approximations were constructed for the solution of a singularly perturbed system of differential equations in the especially critical case when the eigenvalues do not have zeros on the considered domain.

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