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INVERSE PROBLEMS FOR FRACTIONAL SCHRÖDINGER AND SUBDIFFUSION EQUATIONS

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Abstract. *The inverse problems of determining the right-hand side of the Schrödinger and the sub-diffusion equations with the fractional derivative is considered. In the problem 1, the time-dependent source identification problem for the Schrödinger equation , in a Hilbert space* H *is investigated. To solve this inverse problem, we take the additional condition* $B[u(\cdot,t)]=\psi(t)$ *with an arbitrary bounded linear functional* B. *In the problem 2, we consider the subdiffusion equation with a fractional derivative of order* $\rho\in(0,1]$, *and take the abstract operator as the elliptic part. The right-hand side of the equation has the form* $g(t)f$, *where* $g(t)$ *is a given function and the inverse problem of determining element* f *is considered. The condition* $u(t_0)=\psi$ *is taken as the over-determination condition, where* t_0 *is some interior point of the considering domain and* ψ *is a given element. Obtained results are new even for classical diffusion equations. Existence and uniqueness theorems for the solutions to the problems under consideration are proved.*

Key words*: Schrödinger and subdiffusion equation, equation, the Caputo derivatives, Fourier method.*

ОБРАТНЫЕ ЗАДАЧИ ДЛЯ ДРОБНЫХ УРАВНЕНИЙ ШРЕДИНГЕРА И СУБДИФФУЗИИ

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Аннотация. *Рассмотрены обратные задачи определения правой части уравнения Шредингера и уравнения субдиффузии с дробной производной. В задаче 1 исследуется нестационарная задача идентификации источника для уравнения Шрёдингера , в гильбертовом пространстве* H. *Для решения этой обратной задачи возьмем дополнительное условие* $B[u(\cdot,t)]=\psi(t)$ *с произвольным ограниченным линейным функционалом* B. *В задаче 2 мы рассматриваем уравнение субдиффузии с дробной производной порядка* $\rho\in(0,1]$, *а в качестве эллиптической части берем абстрактный оператор. Правая часть уравнения имеет вид* $g(t)f$, *где* $g(t)$ *- заданная функция и рассматривается обратная задача определения элемента* f. *В качестве условия переопределенности принимается условие* $u(t_0)=\psi$, *где* t_0 *- некоторая внутренняя точка рассматриваемой области,* ψ *- заданный элемент. Полученные результаты являются новыми даже для классических уравнений диффузии. Доказаны теоремы существования и единственности решений рассматриваемых задач.*

Ключевые слова: *уравнение Шредингера и субдиффузии, уравнение, производные Капуто, метод Фурье.*

Introduction

The fractional integration of order $\sigma<0$ of the function $h(t)$ defined on $[0,\infty)$ has the form (see, [1]):

$$J_t^{\sigma} h(t) = \frac{1}{\Gamma(-\sigma)} \int_0^t \frac{h(\xi)}{(t-\xi)^{\sigma+1}} d\xi, \quad t > 0,$$

provided the right-hand side exists. Here $\Gamma(\sigma)$ is Euler's gamma function. Using this definition one can define the Caputo fractional derivative of order ρ,

$$D_t^{\rho} h(t) = J_t^{\rho-1} \frac{d}{dt} h(t).$$

If we first integrate and then differentiate, then we get the Riemann-Liouville derivative.

Let H be a separable Hilbert space. Let $A: H \to H$ be an arbitrary unbounded positive selfadjoint operator in H.

Let τ be an arbitrary real number. We introduce the power of operator A, acting in H according to the rule

$$A^{\tau} h = \sum_{k=1}^{\infty} \lambda_k^{\tau} h_k v_k.$$

Obviously, the domain of definition of this operator has the form

$$D(A^{\tau}) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau} | h_k |^2 < \infty\}.$$

For elements $h \in D(A^{\tau})$ we introduce the norm:

$$|| h ||_{\tau}^2 = \sum_{k=1}^{\infty} \lambda_k^{2\tau} | h_k |^2 = || A^{\tau} h ||^2$$

and together with this norm $D(A^{\tau})$ turns into a Hilbert space.

Problem 1. Let $\rho \in (0,1)$ be a fixed number. Consider the following Cauchy problem

$$\begin{cases} iD_t^{\rho} u(t) + Au(t) = p(t)q + f(t), & 0 < t \le T, \\ u(0) = \varphi, \end{cases} \tag{1.1}$$

where a part of the source function $p(t)$ is a scalar function, $f(t) \in C(H)$ and $\varphi, q \in H$ are known elements of H.

To solve this time-dependent source identification problem one needs an extra condition. Following the papers of A. Ashyralyev et al. [2] we consider the additional condition in a rather general form:

$$B[u(t)] = \psi(t), \quad 0 \le t \le T, \tag{1.2}$$

where $B: H \to R$ is a given bounded linear functional, and $\psi(t)$ is the given scalar function. We call the Cauchy problem (1.1) together with additional condition (1.2) the inverse problem.

Problem 2. Let $\rho \in (0,1]$ be a fixed number. Consider the Cauchy problem

$$\begin{cases} D_t^{\rho} u(t) + Au(t) = g(t) f, & 0 < t \le T, \\ u(0) = \varphi. \end{cases} \tag{1.3}$$

Here $\varphi, f \in H$ are known elements of H and a part of the source function $g(t)$ is a scalar function.

To solve the inverse problem of determining the right-hand side of the equation, we use the following additional condition:

$$u(t_0) = \psi, \tag{1.4}$$

where t_0 is a given fixed point of the segment $(0, T]$.

Main results for the Problem 1

Theorem 1. Let $Bq \neq 0$, $\varphi \in H$ and $D_t^{\rho}\psi(t) \in C[0,T]$. Further, let $\epsilon \in (0,1)$ be any fixed number and $q \in D(A^{1+\epsilon})$ and $f(t) \in C([0,T]; D(A^{\epsilon}))$. Then the inverse problem has a unique solution $\{u(t), p(t)\}$.

Theorem 2. Let assumptions of Theorem 1 be satisfied and let $\varphi \in D(A)$. Then the solution to the inverst problem obeys the stability estimate

$$\| D_t^{\rho} u \|_{C(H)} + \| Au \|_{C(H)} + \| p \|_{C[0,T]} \leq C_{\rho,q,B,\epsilon} \Big[\| \varphi \|_1 + \| \psi \|_{C[0,T]} + \max_{0\leq t\leq T} \| f(t) \|_{\epsilon} \Big],$$

where $C_{\rho,q,B,\epsilon}$ is a constant, depending only on ρ, q, B and ϵ.

Similar results hold for the Riemann-Liouville fractional derivative.

Main results for the Problem 2

Lemma 1. Let $\rho = 1$, $g(t) \in C^1[0,T]$ and $g(t_0) \neq 0$. Then there exists a number k_0 such that, starting from the number $k \geq k_0$, the following estimates hold:

$$\frac{C_0}{\lambda_k} \leq | b_{k,1}(t_0) | \leq \frac{C_1}{\lambda_k},$$

where

$$b_{k,1}(t_0) = \int_0^{t_0} e^{-\lambda_k s} g(t_0 - s) ds$$

and constants C_0 and $C_1 > 0$ depend on k_0 and t_0.

Lemma 2. Let $\rho \in (0,1)$, $g(t) \in C^1[0,T]$ and $g(0) \neq 0$. Then there exist numbers $m_0 > 0$ and k_0 such that, for all $t_0 \leq m_0$ and $k \geq k_0$, the following estimates hold:

$$\frac{C_0}{\lambda_k} \leq | b_{k,\rho}(t_0) | \leq \frac{C_1}{\lambda_k},$$

where

$$b_{k,\rho}(t_0) = \int_0^{t_0} g(t_0 - s) s^{\rho-1} E_{\rho,\rho}(-\lambda_k s^{\rho}) ds$$

and constants C_0 and $C_1 > 0$ depend on m_0 and k_0.

Let $\mathbb{N} = K_{\rho} \cup K_{0,\rho}$, where $\mathbb{N}$ is the set of all natural numbers. K_{ρ} and $K_{0,\rho}$ are sets such that: if $b_{k,\rho}(t_0) \neq 0$, $k \in K_{\rho}$, otherwise, if $b_{k,\rho}(t_0) = 0$, then $k \in K_{0,\rho}$.

Theorem 3. Let $\rho \in (0,1]$, $\varphi \in H$, $\psi \in D(A)$. Moreover let function $g(t) \in C[0,T]$ and $g(t) \neq 0$, $t \in [0,T]$. Then there exists a unique solution of the inverse problem (1.3)-(1.4).

Theorem 3 proves the existence and uniqueness of a solution to the inverse problem (1.3)-(1.4) under condition $g(t) \in C[0,T]$ and $g(t) \neq 0$, $t \in [0,T]$, i.e., $g(t)$ does not change sign. Article [3], Example 3.1, shows the non-uniqueness result if $g(t)$ changes its sign. It is proved that if function $g(t)$ does not change sign, then the solution of the inverse problem is unique. Naturally, questions arise: if $g(t)$ changes sign, is uniqueness always violated? What can be said about the existence of a solution? How many solutions can there be?

It should be emphasized that the answers to these questions were not known even for the classical diffusion equation (i.e. $\rho = 1$).

Lemmas 1 and 2 proved above allow us to answer these questions. Let us formulate the corresponding result.

Theorem 4. Let $\varphi \in H$, $\psi \in D(A)$. Further, we will assume that for $\rho = 1$ the conditions of Lemma 1 are satisfied, and for $\rho \in (0,1)$, the conditions of Lemma 2 are satisfied and t_0 is sufficiently small. If set $K_{0,\rho}$ is empty, i.e. $b_{k,\rho}(t_0) \neq 0$, for all k, then there exists a unique solution of the inverse problem (1.3)-(1.4). If set $K_{0,\rho}$ is not empty, then for the existence of a solution to the inverse problem, it is necessary and sufficient that the following conditions

$$\psi_k = \varphi_k E_\rho(-\lambda_k t_0), \quad k \in K_{0,\rho}, \tag{3.1}$$

be satisfied. In this case, the solution to the problem (1.3)-(1.4) exists, but is not unique.

Remark. For conditions (3.1) to be satisfied, it suffices that the following orthogonality conditions hold:

$$\varphi_k = (\varphi, v_k) = 0, \psi_k = (\psi, v_k) = 0, k \in K_{0,\rho}.$$

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