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The influence of small perturbation on phenomenon of delayed loss of stability

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Abstract. The study of solutions to singularly perturbed problems remains relevant, as many mathematical models in technical and natural sciences are described by such differential equations. Despite existing research, there remains a need for a more in-depth analysis and further study of the in-fluence of small perturbations on the phenomenon of delayed loss of stability. The aim of this study was to examine the influence of a small perturbation on the phenomenon of delayed loss of stability, as well as to justify the limit transition confirming the convergence of solutions of the perturbed and unperturbed problems. To achieve this goal, analytical methods were employed, including the level lines method and methods for selecting descending integration paths, which made it possible to rigorously substantiate the limit transitions between the perturbed and unper-turbed problems. The study established that in the absence of a small perturbation, the phenomenon of delayed loss of stability persists regardless of the location of the zeros of the eigenvalues whether on the real axis or in the complex plane. In the presence of a small perturbation, the situation changes: if the eigenvalues have zeros on the real axis, the delay phenomenon does not occur. However, if the zeros are located in the complex plane, the delay is observed only over a limited time interval. In the case where the eigenvalues have poles, the small perturbation does not affect the presence of the phenomenon persists in all cases. Thus, the influence of a small perturbation on the delayed loss of stability depends significantly on the nature of the eigenvalues. It was also substantiated that, under certain conditions on the small perturbation, convergence of solutions is ensured when transitioning from the perturbed problem to the unperturbed one. The results of the study provide a justification for the existence and nature of delayed loss of stability in broader functional spaces, which is important for applied problems in modelling unstable processes

Keywords: small parameter; limit transition; eigenvalues; stability of solutions; integral curves

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Introduction

The phenomenon of delayed loss of stability plays an important role in the theory of singularly perturbed first-order differential equations during changes in stability conditions. The stable and unstable intervals of the solution to singularly perturbed first-order differential equations are determined by the real parts of the eigenvalues. The identification of the negative interval of the real part of the eigenvalue integral justifies the emergence of the phenomenon of delayed loss of stability. Such a case occurs when the zeros of the eigenvalues lie in the complex plane. When they lie on the real line, the question arises about the presence of the extended loss of stability effect and the nature of the system's response to small perturbations. Given the insufficient degree of study of such cases, the presented research is of high relevance.

When the limit equation has turning points, the work of A.G. Eliseev [1] studied the construction of an asymptotic solution of the linear Cauchy problem with a weak turning point of the limit operator using the Lomov regularisation method. The main singularities of this problem are presented in explicit form. Estimates in terms of ε are provided, describing the behaviour of the singularities as $\varepsilon \to 0$. Asymptotic convergence of the regularised series has been proven. The obtained results are illustrated by an example. Small perturbations and the case of stability change are not considered in this work.

In the work of P. Kaklamanos *et al*. [2], an autonomous singularly perturbed system with two fast and one slow variable was considered, in which the linearisation of the fast variable subsystems has intersecting or closely spaced eigenvalues. It has been shown that such a spectral structure leads to the emergence of the delayed loss of stability effect, where the system's trajectory remains near an unstable equilibrium longer than expected. The authors proposed generalised entry-exit relations formulas that allow for a quantitative description of the nature and duration of this delayed transition, including the influence of eigenvalue crossings on the geometry of the phase space.

Sometimes, when the zeros of the eigenvalues lie on the real line, it is possible to determine stable and unstable intervals, but the integral taken over the eigenvalues does not allow for the determination of a stable interval. In this case, the point of stability condition change is chosen as the initial point, which allows for the determination of a stable region for conducting the study. A.A. Akmatov *et al.* [3] noted this case in their work.

S.K. Karimov *et al*. [4] studied the case when the zeros of the eigenvalues lie on the real axis and small perturbations are present. Here, the eigenvalues satisfy all the conditions for determining stable and unstable intervals. Moreover, on the complex plane, the lines passing through the points of stability change divide the region into four parts. The choice of a descending integration path connecting the initial and final points within the solution domain of the problem is accompanied by certain difficulties. These difficulties are resolved by means of parallel lines, which are well known in projective geometry. However, in this case, there are certain nuances that require explanation. Moreover, it does not fully cover the influence of a small perturbation on the solution of the singular problem.

In the work of A.G. Eliseev [5], based on the regularisation method of S.A. Lomov, an asymptotic solution of the singularly perturbed Cauchy problem for a parabolic equation with the presence of a strong turning point is constructed. The regularisation method allows obtaining a uniform asymptotic solution of the problem on the entire real axis. The idea of the work is based on previously developed methods for solving the singularly perturbed Cauchy problem in the case of a simple turning point of the limiting operator with a natural exponent. However, the case of a singularly perturbed problem with a small perturbation is not considered in this work.

It is known that the zeros of eigenvalues located on the complex plane determine the delay time and the occurrence of the phenomenon of delayed loss of stability. A similar effect was also considered in the context of the FitzHugh-Nagumo model, where a delay in the transition from the steady state to the oscillatory regime is observed during a slow passage through the Hopf bifurcation point. As shown in one of the studies by S.M. Baer *et al*. [6], the system enters oscillations at a parameter value significantly exceeding the critical one, indicating the presence of memory and delay effects not accounted for in the classical bifurcation analysis.

M.N. Nurmatova [7] studied the case when the eigenvalues were complex conjugates, which means that the zeros of the eigenvalues belong to the complex plane. One of the features of the study is the change in the zeros of the eigenvalues, which, in turn, leads to a change in the delay time. Only the smallest zeros of the eigenvalues affect the obtained estimate. A small perturbation is present, and the eigenvalues are complex conjugates, which indicates the existence of the phenomenon of delayed loss of stability.

The aim of the study was to investigate the influence of a small perturbation on the behaviour of the solutions of the problem, as well as to examine its effect on the phenomenon of delayed loss of stability. An important aspect of the work was the justification of the limiting transition, which confirmed the convergence of the solutions of the perturbed and unperturbed problems.

Materials and Methods

Within the framework of the study, the problem of analysing the characteristics of solutions to singularly perturbed differential equations of the following form was considered:

$$\varepsilon x'(t,\varepsilon) = a(t)x(t,\varepsilon) + \varepsilon[f(t) + g(t,x)], \qquad (1)$$

$$x(0,\varepsilon) = x^0(\varepsilon),\ |x^0(\varepsilon)| = O(\varepsilon), \qquad (2)$$

where $0 < \varepsilon$ – is a small parameter, $g(t, x(t, \varepsilon))$ – is an analytic function of two variables, $t \in T$ is a finite or infinite domain. For example, $g(t, x(t, \varepsilon))$ may turn out to be polynomials in the variable x with analytic coefficients on the domain T.

Definition. The expression $\varepsilon f(t)$, where $0 < \varepsilon$ is a small parameter, is called a small perturbation.

The unperturbed equation $a(t)\ \tilde{x}(t,\ 0) = 0$ has the zero solution $\tilde{x}(t) = 0$. In the course of the study, it is necessary to prove the limiting equality:

$$\lim_{\varepsilon \to 0} x(t,\varepsilon) = \tilde{x}(t), \text{ as } t \in T. \qquad (3)$$

Let the following conditions be satisfied:

I) $\varepsilon f(t) = 0$ and $a(t) = \alpha(t)$ or $a(t) = \alpha(t) + i\beta(t)$.
II) $\varepsilon f(t) \neq 0$ and $a(t) = \alpha(t)$ or $a(t) = \alpha(t) + i\beta(t)$.

Here $a(t) = \alpha(t)$, defines the stable and unstable intervals of the eigenvalues, indicating the transition point from stability to instability. By analysing the real part of the eigenvalue $a(t) = \alpha(t) + i\beta(t)$ and solving it as an equation, one can determine the stable and unstable intervals, as well as the transition points from stability to instability.

To analyse the stable interval on the numerical axis, the method of integration over eigenvalues, previously determined on stable and unstable intervals, was applied. This point was one of the main conditions of the problem solution study. At the next stage, the method of Lagrange variation was applied to obtain an analytical representation of the solutions of problems (1) and (2). This method allowed expressing the solution in an integral form suitable for subsequent estimation and theoretical analysis within the framework of the posed problem. To estimate the solution presented in analytical form, a descending integration method was chosen. Accordingly, the level set method [8] or the stationary phase method [9], or simply an appropriate descending integration method, were used. Within the framework of the study of boundary layers, the method [10] related to the investigation of transition curves was applied.

The obtained analytical expression was solved using the method of successive approximations, traditionally applied in the theory of differential equations. The choice of this method was determined by its ability to provide the construction of an approximate solution with the subsequent possibility of obtaining an exact estimate. Majorant series were used to demonstrate the convergence of the obtained estimates. This, in turn, facilitated the process of achieving the corresponding convergence of the estimates.

At the next stage, it was planned to prove the uniqueness of the solution by the method of contradiction. However, this step was not carried out within the framework of the work, as it was reduced to standard formal procedures. Nevertheless, the structure of the study was maintained in accordance with the generally accepted approach.

Results and Discussion

Let condition I be satisfied. However, its fulfillment did not eliminate the necessity of solving the differential equation using the Lagrange method, since the problems defined by equalities (1) and (2) were nonlinear. Thus, in accordance with the Lagrange method, a homogeneous approximation was identified, corresponding to the singularly perturbed first-order ordinary differential equation given in equality (1). Then:

$$\varepsilon \frac{dx(t,\varepsilon)}{dt} = a(t)x(t,\varepsilon).$$

By separating the variables and integrating, the general solution was obtained: $x(t,\varepsilon) = C\,exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right)$, where C – is an arbitrary constant. Now, by varying C, taking the value $C = C(t)$. As a result, the following was obtained:

$$x(t,\varepsilon) = C(t)\,exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right).$$

Substituting the derivative $x'(t, \varepsilon)$ and the function $x(t, \varepsilon)$ itself into equality (1), performing some transformations, the following was obtained:

$$x(t,\varepsilon) = x^0(\varepsilon) \times$$
$$\times\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right) + \int_{t_0}^{t} g(\tau, x(\tau,\varepsilon))\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right) d\tau. \quad (4)$$

The integral problem defined by equality (4) was solved using the method of successive approximations. This method simplifies calculations, takes into account small perturbations, and constructs an asymptotic approximation of the perturbed problem's solution to the solution of the corresponding unperturbed problem. The successive approximations were defined as follows:

$$x_0(t,\varepsilon) = 0,$$
$$x_1(t,\varepsilon) = x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right),$$
$$x_2(t,\varepsilon) = x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right) +$$
$$+ \int_{t_0}^{t} g(\tau, x_1(\tau,\varepsilon))\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right) d\tau,$$
$$\dots\dots\dots,$$
$$x_n(t,\varepsilon) = x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right) +$$
$$+ \int_{t_0}^{t} g(\tau, x_{n-1}(\tau,\varepsilon))\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right) d\tau,$$
$$(n = 0, 1, 2, \dots).$$

The unperturbed equation had the zero solution $\tilde{x}(t) = 0$. The initial point belonged to the stable segment, therefore, according to the stability criterion, was chosen $|x(t_0, \varepsilon)| = |x^0(\varepsilon)| = O(\varepsilon)$. The study was conducted in the complex plane. Then:

$$u(t_1, t_2) = Re\int_{t_0}^{t_1+it_2} a(s)ds,$$
$$\vartheta(t_1, t_2) = Im\int_{t_0}^{t_1+it_2} a(s)ds,$$

where $t_1, t_2 \in R$. The solutions of problems (1) and (2) were studied in the domain $T = \{(t_1; t_2): u(t_1, t_2) \le 0\}$, which was defined by the function $u(t_1, t_2)$. From equality (4), the following notation was introduced:

$$A(t,\varepsilon) = x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right)$$
and $I(t,\varepsilon) = \int_{t_0}^{t} g(\tau, x(\tau,\varepsilon))\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right) d\tau.$

In the domain T, the estimate of the absolute value $|A(t,\varepsilon)| = \left|x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right)\right| = O(\varepsilon)$, turned out to be valid. This estimate held due to the absolute value of the initial problem $|x(t_0, \varepsilon)| = |x_0(\varepsilon)| = O(\varepsilon)$.

The zeros of the function defined by the integral of the eigenvalues in the complex plane determined the lines. These lines, in turn, marked the boundaries of the domains under study in the complex plane, within which problems (1) and (2) were solved. A boundary layer was formed in a sufficiently small neighbourhood of these boundaries. In this layer, the estimates defined by the function $A(t, \varepsilon)$ became significant. However, since the initial point was chosen within a stable interval, taking into account the condition $|x^0(\varepsilon)| = O(\varepsilon)$, an estimate could be obtained.

To estimate the function $I(t, \varepsilon)$, integration paths were chosen in the domain T. In order for the successive approximations to remain bounded, it was necessary to satisfy the condition $u(t_1, t_2) - u(\tau_1, \tau_2) \le 0$, that is, the integration paths had to be descending from the initial point to the final one. As a result, the first approximation was determined by the estimate $|A(t, \varepsilon)|$, and then for all the $(t_1; t_2) \in T$ following estimate held:

$$|x_1(t, \varepsilon)| \le C\delta(\varepsilon), \qquad (5)$$

where C is an arbitrary constant.

The function $I(t, \varepsilon)$ had values except within the boundary layers. Therefore, it was necessary to follow a descending path of integration. For $\delta(\varepsilon)$, the statement held true $\varepsilon = o(\delta(\varepsilon))$, provided that ε tended to zero. The remaining approximations were determined by the estimate of the function $I(t, \varepsilon)$ in the domain T. A majorant expression in the form of a series was constructed, corresponding to the difference:

$$\sum_{n=1}^{\infty}[x_n(t, \varepsilon) - x_{n-1}(t, \varepsilon)]. \qquad (6)$$

The convergence of successive approximations was studied. Subsequently, positive constants that did not play a significant role in the reasoning were denoted by the same letter C. It is assumed that the following condition was satisfied:

$$|g(t, x) - g(t, \tilde{x})| \le \beta|x - \tilde{x}|, \qquad (7)$$

where $0 < \beta$ – is a certain constant. Then:

$$x_2(t,\varepsilon) - x_1(t,\varepsilon) = \int_l [g(\tau, x_1(\tau,\varepsilon))]\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right) d\tau.$$

The absolute value was determined as follows:

$$|x_2(t,\varepsilon) - x_1(t,\varepsilon)| \le \beta \int_l |x_1(t,\varepsilon)|\, exp\left(\frac{u(t_1,t_2)-u(\tau_1,\tau_2)}{\varepsilon}\right) \cdot$$
$$\cdot\, |d\tau| \le \beta C\sqrt{\varepsilon}\int_l exp\left(\frac{u(t_1,t_2)-u(\tau_1,\tau_2)}{\varepsilon}\right) \cdot |d\tau| \le (C\delta(\varepsilon))^2.$$

The corresponding estimate was obtained

$$|x_2(t, \varepsilon) - x_1(t, \varepsilon)| \le (C_\delta(\varepsilon))^2. \qquad (8)$$

The following expression then took place:

$$|x_n(t, \varepsilon) - x_{n-1}(t, \varepsilon)| \le (C_\delta(\varepsilon))^n, (n = 1, 2,...).$$

The proof of the uniqueness of the solution was carried out similarly to that performed in the work of L.S. Pontryagin & E.F. Mishchenko [11]. The following theorem holds true.

Theorem 1. In the domain T, the problem (1), (2) has a unique solution $x(t, \varepsilon)$, representable in the form $x(t, \varepsilon) = \sum_{n=1}^{\infty}[x_n(t, \varepsilon) - x_{n-1}(t, \varepsilon)]$, and on T the estimate $|x_n(t, \varepsilon) - x_{n-1}(t, \varepsilon)| \le (C_\delta(\varepsilon))^n$, $(n = 1, 2,...)$ holds, where $0 < C$ is a constant number.

Let condition II be satisfied. In this case, the choice of the initial point, the determination of stable and unstable intervals, as well as the definition of the domain T for solving problem (1), (2), were carried out in full analogy with case I. The problem (1), (2) in this case was also solved using the Lagrange variation method. Without repeating the steps carried out in section I, by incorporating the features of this case, when the function $f(t)$ was nonzero, the solution could be written as follows:

$$x(t,\varepsilon) = x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right) +$$
$$+ \int_{t_0}^{t}[f(\tau) + g(\tau, x)]\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right) d\tau. \qquad (9)$$

The problem posed in (9) was again solved using the method of successive approximations. The successive approximation was written for the case when the function $f(t)$ was nonzero. Thus, the following expression was obtained:

$$x_0(t, \varepsilon) = 0,$$
$$x_1(t,\varepsilon) = x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right) +$$
$$+ \int_{t_0}^{t} f(\tau)\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right) d\tau,$$
$$x_2(t,\varepsilon) = x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right) +$$
$$+ \int_{t_0}^{t} g(\tau, x_1(\tau,\varepsilon))\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right) d\tau,$$
$$\dots\dots\dots,$$
$$x_n(t,\varepsilon) = x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right) +$$
$$+ \int_{t_0}^{t} g(\tau, x_{n-1}(\tau,\varepsilon))\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right) d\tau,$$
$$(n = 0, 1, 2,...).$$

In this case, since the function $f(t)$, which defines the nonhomogeneous part, was nonzero, the integration path at the first approximation was chosen so that it decreased from the initial point to the endpoint t. As a result, a value was obtained that represented an infinitesimal of a lower order compared to the estimate given in Theorem 1 and depended on the special points of the eigenvalues. Thus, in the domain T, the following estimate was obtained:

$$|x_1(t, \varepsilon)| \leq C\gamma(\varepsilon), \quad (10)$$

where C is an arbitrary constant. As can be seen, the estimate $\gamma(\varepsilon)$, defined by equation (10), tended to zero as the small parameter approached zero. However, the order of this convergence was lower than the order of the estimate in Theorem 1. This is because in that case, the first approximation was determined by the function $A(t, \varepsilon)$, whereas here it was determined by the function $x(t, \varepsilon)$. Since the solution $x(t, \varepsilon)$ included a nonhomogeneous part, the special points of the eigenvalues influenced the order of the solution estimate. For this reason, the equality $\varepsilon = o(\gamma(\varepsilon))$ held true as the small parameter tended to zero.

To prove the convergence of the estimates obtained through successive approximations, the majorant series (6) was also used in this case. Accordingly, assuming the fulfillment of the equality condition (7), and generally taking absolute values, the following estimate was obtained:

$$|x_n(t, \varepsilon) - x_{n-1}(t, \varepsilon)| \leq (C\gamma(\varepsilon))^n, \ (n = 1, 2, \ldots).$$

The uniqueness of the solution was proved similarly to Theorem 1. The following theorem holds.

Theorem 2. In the domain T, the problem (1), (2) had a unique solution $x(t, \varepsilon)$ representable in the form $x(t,\varepsilon) = \sum_{n=1}^{\infty}[x_n(t,\varepsilon) - x_{n-1}(t,\varepsilon)]$, and the estimate $|x_n(t, \varepsilon) - x_{n-1}(t, \varepsilon)| \leq (C\gamma(\varepsilon))^n$, $(n = 1, 2, \ldots)$ was valid on T, where $0 < C$ is a constant number.

The first theorem was studied in the absence of small perturbations and proved that the phenomenon of delayed loss of stability occurred independently of the singular points of the eigenvalues. The second theorem proved that this phenomenon took place when the singular points of the eigenvalues were located in the complex plane.

In their work, the authors S.F. Iglesias & S. Mirrahimi [12] studied the asymptotic behaviour of solutions of a Lotka-Volterra type parabolic equation with a periodically varying growth coefficient and nonlocal competition. It has been proven that, at large times, the solution converges to a unique periodic solution. At small mutations, the solution concentrates in the form of a single delta function, and the population size changes periodically over time. Moreover, using methods from the theory of Hamilton-Jacobi equations with constraints, a detailed asymptotic characterisation of such behaviour was obtained. For small but nonzero values of mutations, formal approximations of the moments of the population distribution were proposed, which makes it possible to describe the dynamics of its evolution more accurately. The authors also demonstrated how the obtained results can be applied to interpret and predict biological experiments, confirming the significance of mathematical modelling in studying the adaptation of populations in changing environments.

The authors D.A. Tursunov & G.A. Omaralieva [13] considered the Cauchy problem for a first-order ordinary differential equation with a small parameter at the derivative and a singularity at the initial point. A sufficient condition was found, the fulfillment of which leads to the appearance of an intermediate boundary layer in the singularly perturbed problem described by first-order ordinary differential equations. Using the modified boundary function method, a complete asymptotic expansion of the solution in the form of a series in the Erdélyi sense was constructed. The obtained expansion was justified, that is, an appropriate estimate was obtained for the remainder term. The study is devoted to the investigation of the boundary layer structure, for which necessary estimates were also obtained.

In the work of A.S. Ryabenko [14], a study was conducted on problems of evolutionary differential equations with a complex parameter. Of special interest was the behaviour of their solutions for large values of time, as it demonstrated their evolution. A homogeneous ordinary differential equation with a variable coefficient and a complex parameter was considered, to the study of which a wide class of problems of evolutionary differential equations could be reduced. Unlike the considered work, the parameter was complex and was not associated with the highest derivative, which affected the structure and behaviour of the solutions of this differential equation.

In the article by V.I. Uskov [15], the Cauchy problem for a first-order differential equation in a Banach space was considered, containing a small parameter at the highest derivative and an operator term of Fredholm type on the right-hand side. The relevance of the problem associated with a small parameter at the highest derivative is due to the need to model various physical processes, such as the behaviour of viscous flow, deformations of thin plates and shells, as well as supersonic flow around blunt bodies. The existence of a boundary layer has been identified in the work, which significantly affects the solution even under small perturbations. The asymptotic expansion of the solution is constructed using the Vasilieva-Vishik-Lyusternik method in the form of a power series in the small parameter, and its validity is confirmed by mathematical justification. The regular part of the decomposition is formed by means of the equation decomposition method, which involves a sequential reduction of the problem's dimensionality.

K.G. Kozhobekov *et al*. [16] constructed a uniform asymptotic expansion of the solution to the first boundary value problem for a singularly perturbed second-order parabolic equation. The Vishik-Lusternik method is used, as well as the maximum principle and methods of integrating ordinary differential equations. The solution is presented as the sum of the outer solution and several boundary layers that exponentially decay outside these layers. The remainder term is estimated, which confirms the asymptotic nature of the expansion.

The authors J. Penalva *et al*. [17] studied the phenomenon of delay of loss of stability in an autonomous fast-slow system with a piecewise-linear structure and a slowly varying parameter. It is shown that when the eigenvalues of the fast variables cross or approach each other, a delayed Hopf bifurcation occurs, known as the "way-in/way-out" effect. The authors presented the conditions for the occurrence of this phenomenon and described the so-called entry-exit functions, which depend on the initial conditions and the duration of the delay. Applied to a neuronal model (elliptic bursting), it is shown that these mechanisms are robustly realised even in simple piecewise-linear systems.

In the work by J.T. Zhusubaliyev *et al*. [18], a study was conducted on the bifurcation structure related to the stability of oscillations, bistability, and synchronisation of forced oscillations in a relay system with hysteresis. The behaviour of this system was described by a non-autonomous differential equation with a discontinuous right-hand side. The basic properties of this equation were considered first. Then, a method for obtaining the first return map from this vector field was presented, and it was shown that, depending on the parameter values, such a map can be either a diffeomorphism of the circle or a map with discontinuities. An equation has been identified that divides the parameter plane into regions where the map is either smooth and invertible or discontinuous. A detailed analytical and numerical bifurcation analysis has been carried out, explaining the mechanism of transition between stable capture regimes, bistable states, and chaotic dynamics. Moreover, this work allows the system to be considered as a mathematical model of an oscillatory process describing the dynamics of transitions between different operating modes of the relay system.

In their work, S. Karimov & G.M. Anarbaeva [19] investigated the solution of a singularly perturbed problem under changing stability conditions, taking into account critical points that are the endpoints of delay times. This work addresses unresolved problems related to this class of equations. The analysis is carried out in the irregular case when the singular points are located on the boundaries of the domain. The existence of a solution to the problem under conditions of a bounded domain has been proved. Asymptotic expansions of solutions have been constructed, which allows for a deeper understanding of the dynamics of processes in the considered systems. The analysis of the studied works allows us to conclude that the eigenvalues are complex conjugates, and a small perturbation is present; that is, the investigations were not conducted in the absence of a small perturbation.

The work by P.V. Kirichenko [20] is devoted to the development of a regularisation method for singularly perturbed Cauchy problems in which the spectral stability conditions of the limiting operator are violated. The case of a "weak" turning point is considered, in which the eigenvalues coalesce at the initial moment of time. The principles of introducing regularising functions, the regularisation algorithm, and its mathematical justification are presented in detail in the work. An asymptotic solution of arbitrary order with respect to the small parameter is constructed, demonstrating the effectiveness of the method under spectral peculiarities.

Conclusions

Within the framework of the conducted study, the phenomenon of delayed loss of stability in singularly perturbed problems was examined under various configurations of eigenvalues and in the presence or absence of a small perturbation. It has been shown that in the absence of a perturbation, the effect of delayed loss of stability arises regardless of the nature of the spectrum: both when the zeros of the eigenvalues are located in the complex plane and on the real axis. A sufficient condition for the existence of this phenomenon is the fulfilment of certain spectral requirements that do not depend on the specific location of the zeros.

Special attention is given to cases where the eigenvalues contain poles. It has been established that the presence or absence of a small perturbation does not affect the manifestation of the effect – the determining factor remains the structure of the spectrum. In contrast, in the absence of poles and in the presence of a small perturbation, the behaviour of the system is determined not only by the eigenvalues but also by the form of the perturbation. In particular, if the zeros are located on the real axis, the asymptotic closeness of the solutions is preserved only until the loss of stability, which indicates the dominant role of the perturbation itself.

It has been established that the order of the zeros of eigenvalues affects the order of the resulting solution estimates, which is important for constructing a priori bounds and analysing long-term dynamics. In the case of purely imaginary values, the time delay is absent, and the behaviour is determined exclusively by the spectrum.

Thus, the objective of the study has been achieved: the conditions for the emergence of the effect of delayed loss of stability have been characterized, and their relationship with spectral properties and external perturbations has been substantiated. The results obtained are of interest for the further development of stability theory in singularly perturbed problems, especially in more general types of functional spaces.

A promising direction for future work may involve refining the understanding of the influence of perturbations in the complex plane, taking into account the geometry of the spectrum and boundary conditions.

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Conflict of Interest

None.

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Туруктуулуктун жоголушунун тартылышы кубулушуна кичине козголуунун тийгизген таасири

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Аннотация. Сингулярдык козголгон маселелердин чечимин изилдөө бүгүнкү күндө дагы актуалдуу бойдон калууда, анткени техникалык жана табигый илимдердеги көптөгөн математикалык моделдер дал ушул дифференциалдык теңдемелер аркылуу сүрөттөлөт. Учурдагы изилдөөлөргө карабастан, кичине козголуунун туруктуулуктун жоголуусунун тартылышы кубулушунуна тийгизген таасирин тереңирээк талдоо жана андан ары изилдөө зарылдыгы сакталууда. Бул изилдөөнүн максаты кичине козголуунун туруктуулуктун жоголушунун тартылышы кубулушуна тийгизген таасирин изилдөө, ошондой эле козголгон жана козголбогон маселелердин чечимдеринин жакындашуусунун пределдик өтүүсүн негиздөө болуп саналат. Койулган максатка жетүү үчүн аналитикалык ыкмалар колдонулду, алардын ичинде деңгээл сызыктары методу жана кемүүчү интегралдоо жолдорун тандоо ыкмалары бар, бул болсо козголгон жана козголбогон маселелердин ортосундагы пределдик өтүүлөрдү так негиздөөгө мүмкүндүк берди. Жумушта кичине козголуу жок болгондо, туруктуулуктун жоголуусунун тартылышы кубулушу өздүк маанилердин нөлдөрү сан огунда же комплекстүү тегиздикте жайгашканына карабастан сакталары аныкталды. Кичине козголуу бар болгон учурда абал өзгөрөт: эгерде өздүк маанилер сан огунда нөлдөргө ээ болсо, туруктуулуктун жоголуусунун тартылышы кубулушу байкалбайт. Бирок, эгерде нөлдөр комплекстүү тегиздикте жайгашса, туруктуулуктун жоголуусунун тартылышы кубулушу белгилүү бир убакыт аралыгында гана сакталат. Эгерде өздүк маанилер полюстарга ээ болсо, кичине козголуу бул кубулуштун болушуна таасир этпейт жана ал бардык учурда сакталат. Ошентип, кичине козголуу туруктуулуктун жоголушунун тартылышы кубулушуна тийгизген таасири өздүк маанилердин табиятынан олуттуу түрдө көз каранды болору көрүндү. Ошондой эле кичине козголуу үчүн белгилүү бир шарттар аткарылган учурда козголгон маселеден козголбогон маселеге өткөндө алардын чечимдердин жакындашуусу камсыздалаары негизделди. Изилдөөнүн жыйынтыктары туруктуулуктун жоголушунун тартылышынын болушу жана анын мүнөзүн кеңири функционалдык мейкиндиктерде негиздөөгө мүмкүнчүлүк берет. Бул болсо туруксуз процесстерди моделдөөгө байланыштуу прикладдык маселелер үчүн өзгөчө мааниге ээ болот

Негизги сөздөр: кичине параметр; пределдик өтүүлөр; өздүк маани; чечимдин туруктуулугу; интегралдык ийрилер

Влияние малого возмущения на явление затягивания потери устойчивости

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Аннотация. Изучение решений сингулярно возмущённых задач остаётся актуальным, поскольку многие математические модели в технических и естественных науках описываются именно такими дифференциальными уравнениями. Несмотря на имеющиеся исследования, сохраняется необходимость в более глубоком анализе и дальнейшем изучении влияния малого возмущения на явление затягивания потери устойчивости. Целью настоящего исследования являлось изучение влияния малого возмущения на явление затягивания потери устойчивости, а также обоснование предельного перехода, подтверждающего сходимость решений возмущённой и невозмущённой задач. Для достижения поставленной цели были использованы аналитические методы, включая метод линий уровня и методы выбора убывающих путей интегрирования, что позволило корректно обосновать предельные переходы между возмущённой и невозмущённой задачами. В работе установлено, что при отсутствии малого возмущения явление затягивания потери устойчивости сохраняется независимо от расположения нулей собственных значений как на вещественной оси, так и в комплексной плоскости. При наличии малого возмущения ситуация меняется: если собственные значения имеют нули на вещественной оси, явление затягивания не наблюдается. Однако, если нули расположены в комплексной плоскости, затягивание сохраняется лишь на ограниченном временном интервале. В случае, когда собственные значения обладают полюсами, малое возмущение не влияет на наличие данного явления оно сохраняется в любом случае. Таким образом, влияние малого возмущения на затягивание потери устойчивости существенно зависит от природы собственных значений. Также было обосновано, что при выполнении определённых условий на малое возмущение обеспечивается сходимость решений при переходе от возмущённой задачи к невозмущённой. Результаты исследования позволяют обосновать существование и характер затягивания потери устойчивости в более широких функциональных пространствах, что важно для прикладных задач моделирования нестабильных процессов

Ключевые слова: малый параметр; предельный переход; собственные значения; устойчивость решений; интегральные кривые