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INTEGRATION OF A THIRD-ORDER ODE VIA ANALYTICAL AND GEOMETRICAL METHODS

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Abstracts. *Analytical and geometrical methods are applied to integrate an ordinary differential equation of third order. The main objective is to compare both approaches and show the possibilities that each one of them offers in the integration process of the considered equation, specially when not only Lie point symmetries but also generalized symmetries are involved. The analytical method of order reduction by using a generalized symmetry provides the general solution of the equation but in terms of a primitive that cannot be explicitly evaluated. On the other hand, the application of geometrical tools previously reported in the recent literature leads to two functionally independent first integrals of the equation without any kind of integration. In order to complete the integration of the given third-order equation, a third independent first integral arises by quadrature as the primitive of a closed differential one-form. From these first integrals, the expression of the general solution of the equation can be expressed in parametric form and in terms of elementary functions.*

Key words: *Lie symmetries, first-order symmetries, first integrals, involutive distributions.*

1. Introduction.

The Lie symmetry approach for reducing the order and integrating ordinary differential equations (ODEs) is one of the most powerful and used tools available for handling ODEs and their exact solutions. It is well known that an n^{th}-order equation admitting an r-dimensional algebra of Lie point symmetries can be reduced to an $(n-r)^{th}$-order equation in terms of the common differential invariants of the symmetries. Furthermore, if this algebra is solvable then one can recover the solution of the original equation by solving the reduced ODE and carrying out r consecutive quadratures [1, 2, 3, 4, 5]. In this paper, we will refer to these procedures of order reduction or integration of ODEs as analytical (or classical) methods. This method can be extended to use higher-order symmetries to reduce the order of the equation [3, 6, 7], although in this case the calculation of the differential invariants gets more complicated since, in general, it is not possible to obtain a complete set of differential invariants by derivation of lower order invariants. Furthermore, the reduction process in this situation for an n^{th}-order ODE leads to a system of $n-1$ first-order ODEs that is not equivalent to an $(n-1)^{th}$-order ODE.

In the last decade of the past century, P. Basarab-Horwath [8], J. Sherring and G. Prince [9], T. Hartl and C. Athorne [10] and M. A. Barco and G. Prince [11] obtained powerful geometrical results regarding the integration by quadratures of involutive distributions of vector fields. Such results are based on the concept of solvable structure, an object that generalizes the notion of solvable Lie algebra of symmetries of an involutive distribution. These geometrical results can be applied to integrate ODEs by quadratures by considering the involutive distribution generated by the vector associated to the equation and determining a solvable structure for it. Remarkably, the vector fields involved in a solvable structure are not necessarily Lie point nor generalized symmetries of the equation. Once a solvable structure is known, the equation can be integrated by quadratures by following a procedure that has been studied, applied and generalized by many authors over the years since its introduction [10, 11, 12, 13].

In particular, in the general setting of an n-dimensional manifold, it was developed a procedure to integrate by quadratures involutive distributions of vector fields of dimension r admitting an $(n-r)$-dimensional solvable structure ([8, prop. 3], [9, prop. 4.6 and 4.7]). This method works by constructing $n-r$ closed differential 1-forms which belong to the annihilator of the distribution and which can be integrated successively. This procedure generalizes a classical result known as Lie-Bianchi's Theorem [14, th. 1.7.2], which uses solvable Lie algebras of symmetries instead of solvable structures. We will refer to the application of these last procedures to the involutive distribution associated to an ODE as the geometrical method of integration of the equation.

In the present paper we aim to perform a comparison of the results obtained when both the analytical and the geometrical methods are applied to integrate a third-order ODE. The considered equation can be found in [3, eq. (3.245)], where the authors proved that the equation admits three Lie point symmetries and seven first-order symmetries. The analytical method of reduction of order is applied using one of the first-order symmetries. We will review this procedure and its flaws, and we will see how the geometrical method can give us some advantages in this situation.

The work is organized as follows. In Section 2 the basics definitions and results regarding the method of solvable structures to integrate involutive distributions of vector fields are briefly introduced. In Section 3, with the aim of being self-contained, we describe the reduction of order of ODEs via Lie point symmetries and generalized symmetries, as well as the application of the geometrical methods in the particular case of ODEs. In Section 4, we introduce the ODE under study and the Lie point and generalized symmetries admitted by the equation. We also review the analytical method of reduction of order described in [3], showing some of the problems that may appear when using generalized symmetries. As a consequence, an implicit expression for the general solution of the ODE in terms of a primitive that cannot be evaluated is obtained. In Section 5 we apply geometrical tools to the study of the third-order ODE. It is proved that at least two functionally independent first integrals can be obtained without any kind of integration. A third functionally independent first integral can be obtained by quadrature as a primitive of certain 1-form. The geometric approach allows us to give the general solution of the ODE in parametric form and expressed in terms of elementary functions, greatly improving the results obtained by the analytical method.

2. Preliminaries.

Consider an n-dimensional differentiable manifold M. Given a connected, open set $U \subseteq M$, the real vector space of smooth functions defined on U will be denoted as $\mathcal{C}^\infty(U)$. The $\mathcal{C}^\infty(U)$-module of smooth vector fields defined on U will be denoted as $\mathfrak{X}(U)$. The $\mathcal{C}^\infty(U)$-module of differential p-forms in U will be denoted as $\Omega^p(U)$ [15, def. 2.15], and the exterior algebra will be denoted as $\Omega^*(U)$ [15, def. 2.14]. The exterior product of differential forms will be denoted by $\wedge$. The contraction of a p-form ω by a vector field X [16, pg. 72 (d)] will be written as $i_X\omega$, while the exterior derivative of a p-form ω [16, pg. 70 (b)] will be represented by $d\omega$.

A collection of r vector fields $A_1, \dots, A_r \in \mathfrak{X}(U)$ will be pointwise linearly independent on U (or simply independent) if the vectors $A_1(p), \dots, A_r(p)$ are linearly independent for each $p \in U$. The same applies for a collection of r differential p-forms.

The $\mathcal{C}^\infty(U)$-module generated by $A_1, \dots, A_r \in \mathfrak{X}(U)$ will be called the r-dimensional **distribution** $\mathcal{D}$ generated by the vector fields (see [15, def. 1.56], [13, sec. 2.2]) and will be denoted by

$$\mathcal{D} = \langle A_1, \dots, A_r \rangle. \quad (1)$$

We will say that $X \notin \mathcal{D}$ or that X is transversal to $\mathcal{D}$ if the vector fields $X, A_1, \dots, A_r$ are pointwise linearly independent in U.

An r-dimensional distribution $\mathcal{D}$ is said to be **involutive** if it is closed under the Lie bracket [15, def. 1.56]. This condition guarantees, via the well-known Frobenius theorem, the local existence of $n-r$ functionally independent first integrals $I_1, \dots, I_{n-r} \in \mathcal{C}^\infty(U)$, in which case

$$N = \{p \in U: I_j(p) = C_j, j = 1, \dots, n-r\} \quad (2)$$

are integral manifolds of $\mathcal{D}$ for $C_1, \dots, C_{n-r} \in \mathbb{R}$ [15, th. 1.60]. Nevertheless, Frobenius theorem does not provide a procedure to compute these first integrals. In order to find them, the concept of symmetry of a distribution [9, p. 441] is useful, as we will shortly see:

Definition 1. Let $U \subseteq M$ be an open set, and $\mathcal{D}$ an r-dimensional distribution. A vector field Y is a **symmetry** of $\mathcal{D}$ if for every $A \in \mathcal{D}$ it is $[Y, A] \in \mathcal{D}$. A symmetry of $\mathcal{D}$ is called non-trivial if it is transversal to $\mathcal{D}$.

The set of symmetries of $\mathcal{D}$ will be denoted by $\mathrm{Sym}\,\{\mathcal{D}\}$. Using Jacobi's identity, it can be proved that $\mathrm{Sym}\,\{\mathcal{D}\}$ is a Lie algebra, that is, it is a real vector space and $[X, Y] \in \mathrm{Sym}\,\{\mathcal{D}\}$ whenever $X, Y \in \mathrm{Sym}\,\{\mathcal{D}\}$. Nevertheless, in general, $\mathrm{Sym}\,\{\mathcal{D}\}$ is not a $\mathcal{C}^\infty(U)$-module, since the product of a symmetry by a smooth function which does not vanish on U generally is not a symmetry.

The knowledge of an $(n-r)$-dimensional solvable Lie algebra of symmetries of an r-dimensional involutive distribution allows the computation of $n-r$ functionally independent first integrals of the distribution by quadratures alone, a result known in the literature as Lie-Bianchi's theorem [14, th. 1.7.2].

The next concept generalizes the notion of solvable Lie algebra of symmetries of an r-dimensional distribution [8, def. 4]:

Definition 2. Let $\mathcal{D} = \langle A_1, \dots, A_r \rangle$ be an r-dimensional distribution and let $\{Y_1, \dots, Y_{n-r}\}$ be an ordered set of pointwise linearly independent vector fields on U. We will say that the previous ordered set is a **solvable structure** for $\mathcal{D}$ if:

- $\{Y_1, \dots, Y_{n-r}, A_1, \dots, A_r\}$ are pointwise linearly independent in U.
- Y_{n-r} is a symmetry of $\mathcal{D} = \langle A_1, \dots, A_r \rangle$.
- Y_j is a symmetry of $\mathcal{D}_j = \langle Y_{j+1}, \dots, Y_{n-r}, A_1, \dots, A_r \rangle$ for every $j = 1, \dots, n-r-1$.

Observe that, as we announced, an $(n-r)$-dimensional solvable Lie algebra of symmetries of an r-dimensional distribution is a particular case of a solvable structure, as according to [17, prop 1.23] there exists a basis of the Lie algebra, $Y_1, \dots, Y_{n-r} \in \mathfrak{X}(U)$, such that

$$[Y_i, Y_j] = \sum_{k=1}^{j-1} c_{ij}^k Y_k, \quad c_{ij}^k \in \mathbb{R}, \quad 1 \le i < j \le n-r. \quad (3)$$

The next result, whose proof can be found, for instance, in [9, prop. 4.6 and 4.7] or in [8, prop. 3], allows us to integrate by $n - r$ successive quadratures any r-dimensional distribution admitting a solvable structure. It generalizes Lie-Bianchi's theorem by requiring the knowledge of a solvable structure instead of an $(n-r)$-dimensional solvable Lie algebra of symmetries. Before proceeding let us introduce the following notation: given $\omega_1, \ldots, \omega_r \in \Omega^p(U)$ pointwise linearly independent,

$$\mathcal{I}(\omega_1, \ldots, \omega_r) \quad (4)$$

will be the ideal generated by the previous p-forms undertaking exterior products [16, lemma 2.19 (ii)].

Theorem 1. Let $\mathcal{D} = \langle A_1, \ldots, A_r \rangle$ be an r-dimensional, involutive distribution. Let $\Omega \in \Omega^n(U)$ be a non-zero n-form. Suppose that $\{Y_1, \ldots, Y_{n-r}\}$ is a solvable structure of $\mathcal{D}$ and define the following 1-forms, where the hat denotes omission of the element:

$$\omega_j = \frac{1}{i_{Y_1} \cdots i_{Y_{n-r}} i_{A_1} \cdots i_{A_r} \Omega} \left(i_{Y_1} \ldots \widehat{i_{Y_j}} \ldots i_{Y_{n-r}} i_{A_1} \ldots i_{A_r} \Omega \right), \quad j = 1, \ldots, n-r. \quad (5)$$

Then the previous 1-forms are pointwise linearly independent in U and satisfy

$$d\omega_1 = 0, d\omega_j \in \mathcal{I}(\omega_1, \ldots, \omega_{j-1}), \text{ for } j = 2, \ldots, n-r. \quad (6)$$

Expressions (6) imply the local existence of a function I_1 such that

$$\omega_1 = dI_1. \quad (7)$$

By construction of ω_1 (see equation (5) for $j = 1$), we have that

$$i_{A_j}\omega_1 = i_{Y_k}\omega_1 = 0, \text{ for } j = 1, \ldots, r, k = 2, \ldots, n-r. \quad (8)$$

From (7) and (8) it follows that I_1 is a first integral of the involutive distribution $\mathcal{D}_1$, and in particular, of $\mathcal{D}$. The restriction of ω_2 to the submanifold defined by keeping I_1 constant is closed, because according to (6), $d\omega_2 \in \mathcal{I}(dI_1)$. Therefore, there exists a function I_2 such that, locally,

$$\omega_2 = dI_2 - Y_1(I_2)dI_1. \quad (9)$$

We can continue in this way until we have finally found a complete set of functionally independent first integrals $\{I_1, \ldots, I_{n-r}\}$ of the distribution $\mathcal{D}$. More details and examples on the theory of solvable structures and its generalizations can be consulted in [8, 9, 10, 11, 12, 13] and the references therein.

3. Geometrical and analytical methods of reduction of order for ODEs.

In this section we first review the main aspects of the analytical methods of reduction of order of ODEs via Lie (point and generalized) symmetries [1, 2, 3, 4].

3.1 Symmetry methods for ODEs.

Let us consider an n^{th}-order ordinary differential equation of the form

$$u_n = F(x, u, u_1, \ldots, u_{n-1}), \quad (10)$$

where F is a smooth function defined on an open set $M \subseteq \mathbb{R}^{n+1}$ and

$$u_j = \frac{d^j u}{dx^j}, j = 1, \ldots, n. \quad (11)$$

In what follows, $A \in \mathfrak{X}(M)$ will be the restriction of the total derivative operator

$$D_x = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \cdots + u_{n-1} \frac{\partial}{\partial u_{n-2}} + u_n \frac{\partial}{\partial u_{n-1}} + \cdots \quad (12)$$

to the submanifold defined by equation (10):

$$A = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \cdots + u_{n-1} \frac{\partial}{\partial u_{n-2}} + F \frac{\partial}{\partial u_{n-1}}. \tag{13}$$

It can be checked that the graph of the $(n-1)^{th}$-order prolongation of any solution of (10) is an integral curve of the distribution $\langle A \rangle$. Conversely, any integral curve of this distribution can be locally written as the graph of the $(n-1)^{th}$-order prolongation of a solution of (10) [14, ex. 1.1.2].

Following [2, sec. 3.4], the Lie point symmetries of (10) can be characterized as the vector fields

$$\boldsymbol{v} = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} \tag{14}$$

such that

$$\left[\boldsymbol{v}^{(n-1)}, A\right] = -A(\xi)A, \tag{15}$$

where $\boldsymbol{v}^{(n-1)}$ denotes the $(n-1)^{th}$-order prolongation of $\boldsymbol{v}$ (see [2, sec. 4.1] and [1, th. 2.36]).

It is well-known that the knowledge of a Lie point symmetry (14) of an n^{th}-order ODE (10) leads to reducing the equation to an $(n-1)^{th}$-order ODE plus a quadrature. This can be done through canonical coordinates or differential invariants (see, for instance, [3, sec. 3.3.1 and sec 3.3.2]):

1. Canonical coordinates: let $r = r(x,u)$ and $s = s(x,u)$ be corresponding canonical coordinates for $\boldsymbol{v}$ satisfying $\boldsymbol{v}(r) = 0$ and $\boldsymbol{v}(s) = 1$. Then equation (10) reduces to an $(n-1)^{th}$-order ODE

$$\frac{d^{n-1}z}{dr^{n-1}} = G\left(r, z, \frac{dz}{dr}, \dots, \frac{d^{n-2}z}{dr^{n-2}}\right), \tag{16}$$

where $z = \frac{ds}{dr}$.

In the particular case when $n = 1$, then (16) can be written as

$$\frac{ds}{dr} = G(r), \tag{17}$$

which can be integrated by a single quadrature.

2. Differential invariants: The first step is to find two functionally independent invariants $y = y(x,u)$ and $m = m(x,u,u_1)$ of the first-order prolongation of $\boldsymbol{v}$, through the characteristic equations of $\boldsymbol{v}^{(1)}$ (see [3, eq. (3.102)]). Then by successive derivations

$$m_1 := \frac{dm}{dy} = \frac{A(m)}{A(y)}, \quad m_j := \frac{dm_{j-1}}{dy} = \frac{A(m_{j-1})}{A(y)}, \quad j = 2, \dots, n-1, \tag{18}$$

we obtain invariants for the n^{th}-order prolongation of $\boldsymbol{v}$. Moreover, it can be checked that the invariants $\{y, m, m_1, \dots, m_{n-1}\}$ are functionally independent, i.e., $\{y, m, m_1, \dots, m_{n-1}\}$ is a complete set of invariants for $\boldsymbol{v}^{(n)}$. Then equation (10) can be written in terms of these invariants as an $(n-1)^{th}$-order ODE

$$\Delta(y, m, m_1, \dots, m_{n-1}) = 0, \tag{19}$$

where y is the independent variable and m is the dependent variable. If $m = G(y; C_1, C_2, \dots, C_{n-1})$, where $C_1, C_2, \dots, C_{n-1} \in \mathbb{R}$, denotes the general solution of equation (19), then the general solution of (10) arises from the first-order ODE:

$$m(x,u,u_1) = G(y(x,u); C_1, C_2, \dots, C_{n-1}) \tag{20}$$

which reduces to a quadrature because it admits $\boldsymbol{v}$ as a Lie point symmetry.

When, for $n \geq 2$, the infinitesimals ξ and η of a vector field (14) are allowed to depend on derivatives of u with respect to x up to some order $l \leq n-1$, we get an extension of the notion of symmetry, known in the literature with the name of generalized symmetries [1] (also higher-order symmetries [3] or dynamical symmetries [2]). A generalized vector field [1, def. 5.1]

$$\boldsymbol{v} = \xi(x,u,u_1,\dots,u_i)\frac{\partial}{\partial x} + \eta\big(x,u,u_1,\dots,u_j\big)\frac{\partial}{\partial u} \tag{21}$$

can be prolonged in accordance with the prolongation formula [1, th. 2.36] and generalized symmetries can be characterized through the condition (15).

In the calculation and use of generalized symmetries it is very convenient to consider the evolutionary (or characteristic) form of the generalized vector field (21), which takes the form

$$\boldsymbol{v}_Q = Q\frac{\partial}{\partial u}, \tag{22}$$

where $Q = \eta\big(x,u,u_1,\dots,u_j\big) - \xi(x,u,u_1,\dots,u_i)u_1$ denotes the characteristic of (21). The generalized vector field (21) is a generalized symmetry of equation (10) if and only if its evolutionary representative (22) is [1, prop. 5.5]. This permits to consider generalized symmetries of the form

$$\boldsymbol{v} = \eta(x,u,u_1,\dots,u_l)\frac{\partial}{\partial u}, \quad l \leq n-1, \tag{23}$$

whose prolongations take a particularly simple form:

$$\boldsymbol{v}^{(n)} = \eta\frac{\partial}{\partial u} + \sum_{k=1}^{n} A^k(\eta)\frac{\partial}{\partial u_k}. \tag{24}$$

The determination of a generalized symmetry in the form (23) is done through the condition (15) or equivalently, through the invariance criterion [3, th. 3.5.1-1], which provides a symmetry determining equation for the infinitesimal η (see [3, eq. (3.239)]). In general, it is quite complicated to find solutions for such determining equation. It is usual to try to find some particular solutions by some *ad hoc ansatz*, assuming that η has a special dependency on one or more of its arguments.

The independent variable x is always a zeroth-order invariant of a generalized symmetry of the form (23). A system of higher-order invariants $\{w^1,\dots,w^{n-1}\}$ for $\boldsymbol{v}^{(n)}$ can be determined by solving the characteristic system associated to (24), where x is considered as a constant (see [3, eq. (3.298)]).

The symmetry condition (15) implies that, for $j = 1,\dots,n-1$, $A\big(w^j\big)$ is also an invariant of $\boldsymbol{v}^{(n)}$, that can therefore be expressed in terms of the complete set of invariants $\{x, w^1,\dots,w^{n-1}\}$ in the form

$$A\big(w^j\big) = G^j(x,w^1,\dots,w^{n-1}), \quad j = 1,\dots,n-1. \tag{25}$$

In this way we get a reduction of ODE (10) to a system of $(n-1)$ first-order ODEs:

$$\begin{cases} \dfrac{dw^1}{dx} = G^1(x,w^1,\dots,w^{n-1}), \\ \quad\vdots \\ \dfrac{dw^{n-1}}{dx} = G^{n-1}(x,w^1,\dots,w^{n-1}). \end{cases} \tag{26}$$

The details about this procedure can be consulted, for instance, in [3, sec. 3.5.4] and [2]. As far as we are concerned, there are few references that actually present examples of equations admitting higher-order symmetries and use them to reduce the order of the equation. One of these examples, taken from [3], will be analyzed in the Section 4.

3.2 Geometrical methods of reduction for ODEs.

The problem of reducing or integrating the n^{th}-order ordinary differential equation (10) can be formulated in terms of the geometric notions of symmetry and Frobenius integrability [15]. The vector field (13) associated with equation (10) generates a trivially involutive distribution $\mathcal{D} = \langle A \rangle$, that by Frobenius Theorem [15, prop. 1.59 and th. 1.60] is completely integrable.

The $(n-1)^{th}$-order prolongation of a Lie (point or generalized) symmetry $\boldsymbol{v}$ defines a symmetry of the distribution $\mathcal{D} = \langle A \rangle$ in the sense of Definition 1, because the vector field $Y := \boldsymbol{v}^{(n-1)}$ satisfies relation (15).

If equation (10) admits an n-dimensional solvable symmetry algebra of Lie point or generalized symmetries, the procedure described in Theorem 1 can be used to find by quadratures a complete set $\{I_1, \dots, I_n\}$ of first integrals of equation (10), because such symmetry algebra is a particular case of a solvable structure (see also [9, prop. 5.5]).

In order to do that, since the symmetry algebra is solvable, we can choose a basis such that the $(n-1)^{th}$-order prolongations $Y_1, \dots, Y_n$ satisfy (3). Let $\Omega = dx \wedge du \wedge \cdots \wedge du_{n-1}$ be the volume form and denote

$$\Delta = i_{Y_1} \cdots i_{Y_n} i_A \Omega. \tag{27}$$

Observe that Δ is the determinant formed by the coordinates of the vector fields $A, Y_1, \dots, Y_n$, which are pointwise linearly independent. The corresponding 1-form in (5) for $j = 1$ becomes

$$\omega_1 = \frac{1}{\Delta} i_{Y_2} \cdots i_{Y_n} i_A \Omega. \tag{28}$$

By Theorem 1, ω_1 is closed and hence locally exact. A corresponding primitive I_1 arises by quadrature, and it is a first integral of A. Next, we construct the corresponding 1-form in (5) for $j = 2$. The restriction of such 1-form ω_2 to the submanifold defined by $I_1 = C_1$, where $C_1 \in \mathbb{R}$, is closed, and hence, locally exact. This permits to determine a primitive

$$\widehat{I_2} = \widehat{I_2}(x, u, u_1 \dots, u_{n-1}; C_1) \tag{29}$$

by quadrature. Replacing C_1 by $I_1(x, u, u_1, \dots, u_{n-1})$ in (29), we get a function I_2 satisfying (9), which is a first integral of A. Clearly the process is inductive, and it can be continued until we have calculated a complete system $\{I_1, \dots, I_n\}$ of first integrals for A.

Moreover, when more symmetries than the order of the equation are known, the following result can be really powerful, because it allows to obtain first integrals algebraically, without any kind of integration [9, prop. 5.6]:

Proposition 1. Let $A \in \mathfrak{X}(M)$ be the vector field associated with equation (10). Let $\mathcal{E}$ be an involutive distribution containing A. Suppose that $X, Y \in \mathrm{Sym}\{\mathcal{E}\}$ are transversal to $\mathcal{E}$ and that we can write $Y = \alpha X + Z$ for some $Z \in \mathcal{E}$. Then the function α is a (possibly trivial) first integral of A.

As a natural consequence, the knowledge of more extra symmetries may provide several first integrals without integration [9, cor. 5.7]:

Proposition 2. Let $A \in \mathfrak{X}(M)$ be the vector field associated with equation (10) and assume that $X_1, \dots, X_j$ are independent, non-trivial symmetries of $\mathcal{D} = \langle A \rangle$. If Y is an additional non-trivial symmetry of $\mathcal{D}$ such that $Y = \alpha_1 X_1 + \cdots + \alpha_j X_j + \beta A$, then $\alpha_1, \dots, \alpha_j$ are (possibly trivial) first integrals of A.

In Section 5, we will apply these geometrical tools and results to derive new strategies of integration of a third-order ODE that has been studied in [3] by analytical methods based on generalized symmetries.

4. The ODE and its general solution via the analytical method.

We consider the third-order equation

$$u_3 = 6\frac{u_2^2}{u_1}\left(x\frac{u_2}{u_1} + 1\right). \tag{30}$$

This equation was introduced by G. W. Bluman and S. C. Anco in [3, eq. (3.245)] as an example of how to determine generalized symmetries and use them to reduce the equation.

The corresponding vector field $A \in \mathfrak{X}(M)$ associated to equation (30) becomes

$$A = \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u} + \cdots + u_{n-1}\frac{\partial}{\partial u_{n-2}} + 6\frac{u_2^2}{u_1}\left(x\frac{u_2}{u_1} + 1\right)\frac{\partial}{\partial u_{n-1}}, \tag{31}$$

which is defined on the open set:

$$M = \{(x, u, u_1, u_2) \in \mathbb{R}:\ u_1 \neq 0\}. \tag{32}$$

In the cited reference, the authors prove that (30) admits seven generalized symmetries (of first order) given, in evolutionary form, by

$$\boldsymbol{v}_1 = \frac{1}{u_1}\frac{\partial}{\partial u}, \quad \boldsymbol{v}_2 = \frac{1}{u_1^2}\frac{\partial}{\partial u}, \quad \boldsymbol{v}_3 = xu_1^2\frac{\partial}{\partial u}, \quad \boldsymbol{v}_4 = x^2u_1^4\frac{\partial}{\partial u}$$
$$\boldsymbol{v}_5 = (9x^2u_1^2 - 12xuu_1 + 4u^2)\frac{\partial}{\partial u}, \quad \boldsymbol{v}_6 = \left(3x - \frac{2u}{u_1}\right)\frac{\partial}{\partial u}, \tag{33}$$
$$\boldsymbol{v}_7 = (3x^2u_1^3 - 2xuu_1^2)\frac{\partial}{\partial u}.$$

In addition, in [3] was also proved that equation (30) admits the following three independent Lie point symmetries:

$$\boldsymbol{v}_8 = \frac{\partial}{\partial u}, \quad \boldsymbol{v}_9 = u\,\frac{\partial}{\partial u}, \quad \boldsymbol{v}_{10} = x\frac{\partial}{\partial x}, \tag{34}$$

which span a three-dimensional, solvable Lie algebra because the respective commutations relationships become

$$[\boldsymbol{v}_8, \boldsymbol{v}_9] = \boldsymbol{v}_8, [\boldsymbol{v}_8, \boldsymbol{v}_{10}] = [\boldsymbol{v}_9, \boldsymbol{v}_{10}] = 0. \tag{35}$$

In [3, p. 179-181] the generalized symmetry $\boldsymbol{v}_1$ defined in (33) was used to reduce equation (30) to a system of two first-order ODEs. The first step is to calculate a complete set of second-order differential invariants of $\boldsymbol{v}_1$. As we said in Section 3, x is already a zeroth-order invariant. For the remaining ones, G. W. Bluman and S. C. Anco solved the characteristic equations for the corresponding second-order prolongation (24) of $\boldsymbol{v}_1$. By choosing u_1 as the independent variable, such characteristic system becomes [3, eq. (3.304)]:

$$\begin{cases} \frac{du}{du_1} = 6x\frac{u_2^2}{u_1^2} + 4\frac{u_2}{u_1}, \\ \frac{du_2}{du_1} = -\frac{u_1}{u_2}. \end{cases} \tag{36}$$

After using symmetry methods for system (36), the authors found the following second-order invariants of $\boldsymbol{v_1}$:

$$\mathrm{w}^1 = 2\mathrm{x}\mathrm{u}_1^3 + \frac{\mathrm{u}_1^4}{\mathrm{u}_2}, \quad \mathrm{w}^2 = \mathrm{u} - \frac{1}{2\mathrm{u}_1^2}\mathrm{w}^1 - 2\mathrm{x}\mathrm{u}_1. \tag{37}$$

It can be checked that, in this case, $A(w^1) = A(w^2) = 0$. Therefore, according to (26), the corresponding reduced system of two first-order equations becomes

$$\begin{cases} \frac{dw^1}{dx} = 0, \\ \frac{dw^2}{dx} = 0. \end{cases} \tag{38}$$

System (38) is trivial and its general solution is $w^1 = C_1, w^2 = C_2$, where $C_1, C_2 \in \mathbb{R}$. Substituting the expressions (37) and eliminating u_2, they obtain the first-order ODE

$$2xu_1^3 + (C_2 - u)u_1^2 + \frac{C_1}{2} = 0. \tag{39}$$

Solving this ODE yields the general solution of (30). Equation (39) can be written in explicit form as

$$u_1 = G(x, u; C_1, C_2), \tag{40}$$

and it inherits a Lie point symmetry from the first-order symmetry $\boldsymbol{v}_1$ of (30),

$$\widetilde{\boldsymbol{v}}_1 = \frac{1}{G(x,u;C_1,C_2)}\frac{\partial}{\partial u}. \tag{41}$$

However, working with this symmetry is not really convenient, since the expression $G(x, u; C_1, C_2)$ requires to solve (39) as a cubic equation in u_1. In order to avoid this difficulty, Bluman and Anco determined a new Lie point symmetry for equation (39):

$$\widetilde{\boldsymbol{v}} = x\frac{\partial}{\partial x} + \frac{2}{3}(u - C_2)\frac{\partial}{\partial u}. \tag{42}$$

Applying the method of canonical coordinates, equation (39) is reduced to a quadrature. In particular, we can choose

$$\begin{cases} r(x,u) = -\frac{2}{3}\frac{C_2 - u}{x^{\frac{2}{3}}}, \\ s(x,u) = \ln x, \end{cases} \tag{43}$$

so that

$$\widetilde{\boldsymbol{v}} = \frac{\partial}{\partial s}. \tag{44}$$

Writing now (39) in terms of (r, s), carrying out the quadrature and writing the resulting expression back to the original coordinates, the general solution of (39), and thus of (30), is obtained in implicit form:

$$u = C_2 + \left(\frac{u-C_2}{x^{\frac{2}{3}}}\right) \exp\left(\frac{2}{3} H\left(\frac{u-C_2}{x^{\frac{2}{3}}}\right) + C_3\right), \tag{45}$$

where H is a function such that

$$H'(z) = \frac{6p(z)}{z^2 - 3zp(z) + p(z)^2}, \text{ where } p(z) = \left(z^3 + 3\sqrt{3C_1(27C_1 - 2z^3)} - 27C_1\right)^{\frac{1}{3}}. \tag{46}$$

In the integration procedure that has been applied in this section, several analytical methods based on symmetries have been successively used. First, the generalized symmetry $\boldsymbol{v_1}$ given in (33) has been determined. Second, in order to find second-order differential invariants for $\boldsymbol{v_1}$, symmetry methods have been used to find a particular solution of the characteristic system (36). Luckily, in this example, the reduced system (38) can be trivially integrated; however, in general, additional symmetries might be necessary to solve the reduced system. Third, a new symmetry (42) has been determined in order to solve the first-order ODE (39). Finally, the method of canonical coordinates has been used to integrate equation (39). As a result, the general solution of the ODE has been obtained in (45), although it is expressed in implicit form and in terms of a primitive that cannot be explicitly evaluated (see equation (46)).

In the following section, we investigate if the application of geometrical methods to equation (30) can improve the results that have been obtained so far by using only analytical methods.

5. Integration by geometrical methods.

In this section we apply the geometric tools described in Section 3.2 with the aim of providing a more convenient expression for the general solution of equation (30):

$$u_3 = 6\frac{u_2^2}{u_1}\left(x\frac{u_2}{u_1} + 1\right). \tag{47}$$

As in Section 4, we consider the associated vector field (31),

$$A = \frac{\partial}{\partial x} + u_1\frac{\partial}{\partial u} + \cdots + u_{n-1}\frac{\partial}{\partial u_{n-2}} + 6\frac{u_2^2}{u_1}\left(x\frac{u_2}{u_1} + 1\right)\frac{\partial}{\partial u_{n-1}}, \tag{48}$$

defined on the open set $M \subseteq \mathbb{R}^4$ introduced in (32),

$$M = \{(x, u, u_1, u_2) \in \mathbb{R}\colon u_1 \neq 0\}. \tag{49}$$

In order to achieve our objective of obtaining the general solution of (47) by geometrical methods, we can use some of the Lie point and first-order symmetries of the ODE given in (33) and (34). The integration procedure that will be presented in this section does not need to calculate any differential invariants nor canonical coordinates at all.

First, we observe that among the symmetries (33) and (34) it is possible to select three pointwise independent symmetries satisfying the hypothesis of Theorem 1. This can be done, for instance, by considering the second-order prolongations of the symmetries (34) because, according to (35), they span a three-dimensional, solvable Lie algebra, which is a particular case of a solvable structure of the distribution $\mathcal{D} = \langle A \rangle$. However, since for equation (47) we know an oversupply of symmetries, in the next section we will apply the theoretical results presented in Section 3.2, in order to obtain first integrals algebraically, without any kind of integration (see Proposition 1 and Corollary 1). This will be done by conveniently choosing the symmetries that will be used in the integration process. With this aim, we choose the following symmetries (in the sense of Definition 1) of the distribution $\mathcal{D}$:

$$\begin{aligned} Y_1 &= \boldsymbol{v}_8^{(2)} = \frac{\partial}{\partial u}, \\ Y_2 &= \boldsymbol{v}_9^{(2)} = u\frac{\partial}{\partial u} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}, \\ Y_3 &= \boldsymbol{v}_{10}^{(2)} = x\frac{\partial}{\partial x} - u_1\frac{\partial}{\partial u_1} - 2u_2\frac{\partial}{\partial u_2}, \\ Y_4 &= \boldsymbol{v}_2^{(2)} = \frac{1}{u_1^2}\frac{\partial}{\partial u} - \frac{2u_2}{u_1^3}\frac{\partial}{\partial u_1} - \frac{6u_2^2(2xu_2+u_1)}{u_1^5}\frac{\partial}{\partial u_2}. \end{aligned} \tag{50}$$

It can be checked that the commutator relationships become

$$[Y_1, A] = [Y_2, A] = [Y_3, A] = [Y_4, A] = 0,$$

$$[Y_1, Y_2] = Y_1, \ \ [Y_1, Y_3] = [Y_1, Y_4] = [Y_2, Y_3] = 0, \ \ [Y_2, Y_4] = -3Y_4, \ \ [Y_3, Y_4] = 2Y_4. \tag{51}$$

5.1 Two first integrals without integration.

In this section we aim to apply the theoretical results presented in Section 3.2 in order to obtain two first integrals of A by an algebraic procedure, without any kind of integration.

It can be checked that the set $\{A, Y_1, Y_3, Y_4\}$ is linearly independent on the open set

$$V = \{(x, u, u_1, u_2) \in M\colon\ u_2(3\mathrm{x}\mathrm{u}_2 + \mathrm{u}_1) \neq 0\}. \tag{52}$$

This implies that Y_1, Y_3, Y_4 are pointwise linearly independent and transversal symmetries of the distribution $\mathcal{D} = \langle A\rangle$ on V. Consequently, they can be used as the non-trivial symmetries X_i $(i = 1, 2, 3)$ required in Corollary 1.

Since $\{A, Y_1, Y_3, Y_4\}$ is a basis of $\mathfrak{X}(V)$, any additional non-trivial symmetry of $\mathcal{D}$ can be expressed in terms of Y_1, Y_3, Y_4 and A. For instance, the symmetry Y_2 given in (50) is a transversal symmetry of $\mathcal{D}$ that can be written as follows:

$$Y_2 = -\frac{6xu_1u_2 - 2uu_2 + u_1^2}{2u_2}Y_1 - 2Y_3 + \frac{u_1^3(2xu_2+u_1)}{2u_2}Y_4 + 2xA. \tag{53}$$

As a direct consequence of Corollary 1 we conclude that the following functions, corresponding to the coefficients of Y_1 and Y_4 in (53), are non-trivial first integrals of A:

$$I_1 = 6xu_1 - 2u + \frac{u_1^2}{u_2}, \quad I_2 = 2xu_1^3 + \frac{u_1^4}{u_2}. \tag{54}$$

These functions are defined on the open set

$$U = \{(x, u, u_1, u_2) \in \mathbb{R}^4\colon u_1u_2 \neq 0\} \subseteq M, \tag{55}$$

and it can be checked that $dI_1 \wedge dI_2$ does not vanish on U, so I_1 and I_2 are functionally independent first integrals of A on U.

Since the coefficient of Y_3 in (53) is constant, the application of Corollary 1 by using the symmetries Y_1, Y_2, Y_3, Y_4 only provides two functionally independent first integrals of A. In order to complete the integration of the distribution $\mathcal{D} = \langle A\rangle$, one more functionally independent first integral is required. Although Corollary 1 could be applied by using other sets of symmetries of the equation, there is not a criterion to know *a priori* which ones will produce non-trivial and functionally independent first integrals. In the worst case, it might happen that none of the admitted symmetries gave rise to the remaining first integral. In this situation, an alternative strategy must be followed. In the next subsection we illustrate how Theorem 1 can be applied to overcome this possible obstacle.

5.2 A remaining first integral and the general solution of equation (47).

Besides the first integrals I_1 and I_2 given in (54), one more functionally independent first integral of A is required in order to complete the integration of equation (47). In order to determine such first integral, we first observe that the distribution $\mathcal{E} = \langle Y_3, Y_4, A\rangle$ is involutive by commutator relationships (51). Moreover, since

$$A(I_1) = Y_3(I_1) = Y_4(I_1) = 0, \tag{56}$$

we conclude that I_1 is a first integral of $\mathcal{E}$.

Consider the following local change of variables on the open set U defined in (55),

$$\begin{aligned} \varphi: \quad & U \longrightarrow \varphi(U) \\ & (x, u, u_1, u_2) \longmapsto (u, u_1, u_2, I_1). \end{aligned} \tag{57}$$

By means of the push-forward by φ (see [18, pg. 46]), the vector fields A, Y_4, Y_3 are expressed in terms of local coordinates (u, u_1, u_2, I_1) as follows:

$$\begin{aligned} \widehat{A} &= \varphi_* A = u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + \frac{u_2^2(2uu_2 + I_1 u_2 + 5u_1^2)}{u_1^3} \frac{\partial}{\partial u_2}, \\ \widehat{Y_3} &= \varphi_* Y_3 = -u_1 \frac{\partial}{\partial u} - 2u_2 \frac{\partial}{\partial u_2}, \\ \widehat{Y_4} &= \varphi_* Y_4 = \frac{1}{u_1^2} \frac{\partial}{\partial u} - \frac{2u_2}{u_1^3} \frac{\partial}{\partial u_1} - \frac{2u_2^2(2uu_2 + I_1 u_2 + 2u_1^2)}{u_1^6} \frac{\partial}{\partial u_2}. \end{aligned} \tag{58}$$

We can restrict each one of the previous vector fields to the submanifold defined by the level set $I_1 = C_1$, where $C_1 \in \mathbb{R}$, by substituting I_1 by C_1 in (58). We keep denoting the restricted vector fields by $\widehat{A}$, $\widehat{Y_3}$ and $\widehat{Y_4}$ respectively.

By using (51), it can be checked that $\widehat{Y_3}$ and $\widehat{Y_4}$ span a 2-dimensional, solvable Lie algebra of symmetries of the distribution $\widehat{\mathcal{D}} = \langle \widehat{A} \rangle$. In particular, they generate a solvable structure for $\widehat{\mathcal{D}}$. In order to apply Theorem 1, we consider the non-zero 3-form $\Omega = du \wedge du_1 \wedge du_2$ and construct the corresponding 1-form given by (5) for $j = 1$:

$$\begin{aligned} \omega_1 &= \frac{1}{i_{Y_4} i_{Y_3} i_A \Omega} i_{Y_3} i_A \Omega = \\ &= -\frac{1}{2uu_2 + C_1 u_2 + u_1^2} \left(2u_2 du + \frac{2uu_2 + C_1 u_2 + 3u_1^2}{u_1} du_1 - \frac{u_1^2}{u_2} du_2 \right). \end{aligned} \tag{59}$$

Theorem 1 ensures that (59) is closed and, therefore, locally exact. A primitive of ω_1, and hence a first integral of $\widehat{A}$, can be obtained by quadratures:

$$\widehat{J_3} = \ln\left(\frac{u_2}{(2uu_2 + C_1 u_2 + u_1^2) u_1} \right), \tag{60}$$

Thus, the following function

$$J_3 = \exp(-\hat{J}_3) = \left(\frac{u_1^2}{u_2} + 2u + C_1 \right) u_1 \tag{61}$$

is also a first integral of $\widehat{A}$. Substituting now C_1 by I_1 in (61) and writing the obtained expression back in terms of (x, u, u_1, u_2), we obtain a function $I_3 \in \mathcal{C}^\infty(U)$ given by

$$I_3 = 3xu_1^2 + \frac{u_1^3}{u_2}, \tag{62}$$

which is a first integral of A. It can be checked that I_1, I_2 (defined in (54)) and I_3 are functionally independent on U, since $dI_1 \wedge dI_2 \wedge dI_3$ does not vanish on U.

Therefore, the general solution of ODE (47) can be implicitly defined by equations $I_1 = C_1, I_2 = C_2, I_3 = C_3$, where $C_1, C_2, C_3 \in \mathbb{R}$:

$$\begin{cases} 6xu_1 - 2u + \frac{u_1^2}{u_2} = C_1, \\ 2xu_1^3 + \frac{u_1^4}{u_2} = C_2, \\ 3xu_1^2 + \frac{u_1^3}{u_2} = C_3. \end{cases} \tag{63}$$

In order to obtain a parametric expression for the general solution of ODE (47), we eliminate u_2 from the last equation in (63) and choose $u_1 = t$ as a parameter:

$$\begin{cases} x(t) = \frac{C_3 t - C_2}{t^3}, \\ u(t) = \frac{-C_1 t^2 + 4C_3 t - 3C_2}{2t^2}. \end{cases} \tag{64}$$

We depict the graphs of two of the solutions in Figures 1 and 2, obtained by setting different values to the integration constants.

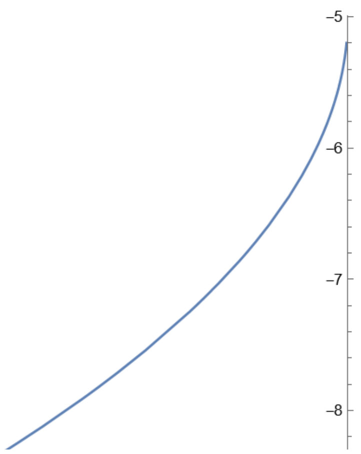

Figure 1:

$C_1 = 10, C_2 = 0, C_3 = -1, 0.5 \le t \le 10.$

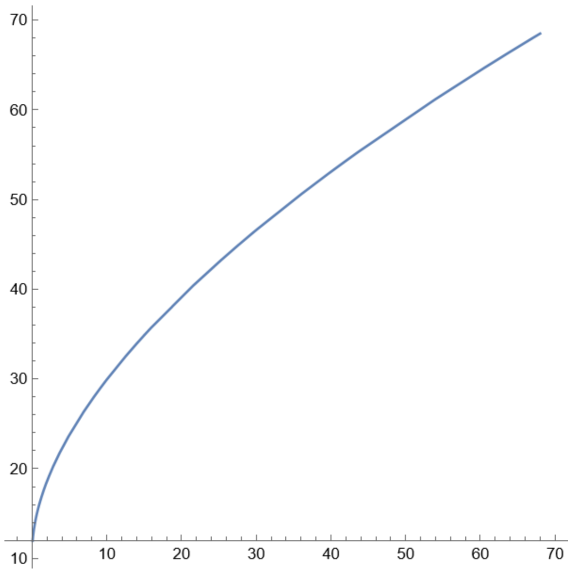

Figure 2:

$C_1 = -21, C_2 = -5, C_3 = 7, 0.5 \le t \le 10.$

6. Concluding remarks.

Different analytical and geometrical methods have been applied in the study of a third-order ODE for which abundant Lie point and generalized symmetries had been previously reported.

Regarding analytical methods, a generalized symmetry of the equation had been used in the previous literature to reduce the given ODE to a system of two first-order ODEs. In this reduction process, additional symmetry methods had been necessary to determine differential invariants for the generalized symmetry. After solving the reduced system, it remains the problem of reconstruction of the solution for the original equation. Although theoretically this can be done by a quadrature, the difficulty of obtaining an explicit expression for the underlying symmetry (41) forced the search of a new symmetry (42). After application of the canonical coordinates method, the implicit general solution (45) was finally obtained. However, this expression involves a primitive that cannot be explicitly evaluated.

In this work we have shown that the application of geometrical methods greatly simplifies the integration of the given third-order ODE. Remarkably, two functionally independent first integrals of the equation have been calculated by simple algebraic manipulations, avoiding the use of differential invariants, canonical coordinates or any kind of integration. Moreover, a remaining first integral has been calculated by quadrature, as a primitive of a 1-form defined in an open set of a three-dimensional space. From the complete system of first integrals of the equation derived by using these geometrical tools we have obtained the general solution of the equation in parametric form (see equation (64)). The obtained solution is given in terms of simple rational expressions, greatly improving the solution (45) derived via the analytical procedures.

It can be concluded that the geometrical approach to integrating ODEs is a powerful alternative to the classical approach of differential invariants or canonical coordinates, specially when there are higher-order symmetries involved. An additional advantage that must be taken into account is that the geometrical methods allow us to use not only symmetries of the associated distribution, i.e., not only prolongations of Lie (point or generalized) symmetries of the equation. In a solvable structure, only the first element must be a symmetry of the associated distribution while, in general, the remaining vector fields are not symmetries of the equation. This fact greatly expands the strategies that can be followed to find exact solutions of differential equations.

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References

1. P. J. Olver. Applications of Lie groups to differential equations. [Text]/ P. J. Olver. // Graduate Texts in Mathematics. Springer US, 2nd edition, 1986.
2. Stephani Differential equations: their solution using symmetries. [Text]/ Stephani and M. MacCallum. // Cambridge University Press, 1989.
3. G. W. Bluman Symmetry and integration methods for differential equations. [Text]/ G. W. Bluman and S. C. Anco //Applied Mathematical Science. Springer New York, 2nd edition, 2002.
4. P. E. Hydon. Symmetry Methods for Differential Equations: A Beginner’s Guide. [Text]/ P. E. Hydon. // Cambridge Texts in Applied Mathematics. Cambridge University Press, 2000.
5. L.V. Ovsiannikov, editor. Group Analysis of Differential Equations. [Text]/ L.V. Ovsiannikov // Academic Press, 1982.
6. P. E. Hydon. Self-invariant first-order symmetries. [Text]/ P. E. Hydon. // Journal of Nonlinear Mathematical Physics, 11(2):233–242, 2004.

7. E. Pucci. First-order symmetries and solutions by reduction of partial differential equations. [Text]/ E. Pucci and G. Saccomandi. // Journal of Physics A: Mathematical and General, 27(1):177–184, 1994.

8. P. Basarab-Horwath. Integrability by quadratures for systems of involutive vector fields. [Text]/ P. Basarab-Horwath. // Ukrainian Math. Zh., 43:1330–1337, 1991.

9. G. Prince Geometric aspects of reduction of order. [Text]/ G. Prince and J. Sherring. // Trans. Amer. Math. Soc., 334 (1):433–453, 1992.

10. M. A. Barco Solvable symmetry structures in differential form applications. [Text]/ M. A. Barco and G. E. Prince. // Acta Applicandae Mathematica, 66:89–121, 2001.

11. T. Hartl. Solvable structures and hidden symmetries. [Text]/ T. Hartl and C. Athorne. // Journal of Physics A: Mathematical and General, 27(10):34–63, 1994.

12. D. C. Ferraioli Local and nonlocal solvable structures in the reduction of odes. [Text]/ D. C. Ferraioli and P. Morando// Journal of Physics A: Mathematical and Theoretical, 42(3):035210, 2008.

13. P. Morando. Reduction by μ–symmetries and σ–symmetries: a Frobenius approach. [Text]/ Journal of Nonlinear Mathematical Physics, 22(1):47–59, 2015.

14. A. Kushner, V. Lychagin and V. Rubtsov. First-order geometry and nonlinear differential equations. Encyclopedia of Mathematics and its Applications. [Text]/ Cambridge University Press, 2007.

15. F. Warner. Foundations of differentiable manifolds and Lie groups. [Text]/ Graduate Texts in Mathematics. Springer, 1971.

16. S. Morita. Geometry of differential forms. Translations of mathematical monographs, [Text]/ Iwanami series in modern mathematics 201. American Mathematical Society, 2001.

17. A. W. Knapp. Lie groups beyond an introduction. [Text]/ Progress in Mathematics. Birkhäuser Boston, 2002.

18. J. M. Lee. Introduction to smooth manifolds. [Text]/ Graduate Texts in Mathematics 218. Springer New York, 2003.