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INITIAL VALUE PROBLEM FOR A NONLINEAR FREDHOLM INTEGRO-DIFFERENTIAL EQUATION OF THIRD ORDER WITH A DEGENERATE KERNEL

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Abstract. *In this paper, it is considered a third order nonlinear Fredholm integro-differential equations with initial value conditions and real parameters. A nonlinear functional-integral equations is derived. Theorem on a uniqueness and existence of the solution of the problem is proved for regular values of parameters. The method of compressing mapping in the space of continuous functions is applied. Continuous dependence on parameters of the solution of initial value problem is studied.*

Key words: *Initial value problem, third order integro-differential equation, unique solvability, real parameters, regular values, dependence on the parameters.*

НАЧАЛЬНАЯ ЗАДАЧА ДЛЯ НЕЛИНЕЙНОГО ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ФРЕДГОЛЬМА ТРЕТЬЕГО ПОРЯДКА С ВЫРОЖДЕННЫМ ЯДРОМ

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Аннотация: *В данной работе рассматривается нелинейное интегро-дифференциальное уравнение Фредгольма третьего порядка с начальными условиями и действительными параметрами. Выводится нелинейное функционально-интегральное уравнение. Доказывается теорема о единственности и существовании решения задачи при регулярных значениях параметров. Применяется метод сжимающих отображений в пространстве непрерывных функций. Изучается непрерывная зависимость от параметров решения начальной задачи.*

Ключевые слова: *Начальная задача, интегро-дифференциальное уравнение третьего порядка, однозначная разрешимость, действительные параметры, регулярные значения, зависимость от параметров.*

Formulation of the problem statement

Integro-differential equations are studied in the works of many mathematics (see, for examples [1-25]). Integro-differential equations with degenerate kernel are studied in the works [26-36].

In this paper we consider the solvability of the initial value problem for a third order integro-differential equation with two real parameter and degenerate kernel. So, we consider the following Fredholm integro-differential equation

$$x'''(t) + \lambda x(t) = \nu \int_0^T K(t,s)\, x(s) ds + F\left(t, \int_0^T G(s)\, x(s) ds \right), \tag{1}$$

where $K(t,s)=\sum_{i=1}^{p}\alpha_i(t)\beta_i(s),\ \ 0<\alpha_i(t),\ \beta_i(s)\in C[0,T],\quad \alpha_i(t)$ and $\beta_i(s)$ are linear independent, $F(t,x)\in C([0,T]\times R),\ \ 0<G(t)\in C[0,T],\ \ T$ is given positive number, $0<\lambda$ is positive finite parameter, ν is nonzero real parameter.

In solving partial integro-differential equation (1), we use the following conditions

$$x(0)=\varphi_1,\ \ x'(0)=\varphi_2,\ \ x''(0)=\varphi_3. \tag{2}$$

Problem statement. To find a function $x(t)\in C[0,T]$, which satisfies integro-differential equation (1) and conditions (2).

Nonlinear integral equations

We use the following denotations

$$f(t)=\nu\sum_{i=1}^{p}\alpha_i(t)\tau_i+F(t,\cdot), \tag{3}$$

$$\tau_i=\int_0^T\beta_i(s)x(s)ds. \tag{4}$$

Then the equation (1) takes the form

$$x'''(t)+\lambda\,x(t)=f(t). \tag{5}$$

The characteristic equation $\sigma^3+\lambda=0$ for the homogeneous equation $x'''(t)+\lambda\,x(t)=0$ has the roots

$$\sigma_1=-\sqrt[3]{\lambda},\quad \sigma_{2/3}=\left(\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right)\sqrt[3]{\lambda}.$$

So, the general solution of the homogeneous equation can be presented as

$$x(t)=A_1x_1(t)+A_2x_2(t)+A_3x_3(t), \tag{6}$$

where $A_k\ (k=1,2,3)$ are yet arbitrary coefficients, which will be determined later,

$$x_1(t)=e^{-\sqrt[3]{\lambda}t},\qquad x_2(t)=e^{\frac{\sqrt[3]{\lambda}t}{2}}\cos\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}t,\quad x_3(t)=e^{\frac{\sqrt[3]{\lambda}t}{2}}\sin\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}t. \tag{7}$$

Taking (6) into account, we search a particular solution of the equation (5) as

$$\tilde{x}(t)=A_1(t)x_1(t)+A_2(t)x_2(t)+A_3(t)x_3(t). \tag{8}$$

In (8) we supposed that $A_k(t)$ are unknown functions. To find these functions we consider the following system of algebraic-differential equations

$$\begin{cases}A_1'(t)x_1(t)+A_2'(t)x_2(t)+A_3'(t)x_3(t)=0,\\ A_1'(t)x_1'(t)+A_2'(t)x_2'(t)+A_3'(t)x_3'(t)=0,\\ A_1'(t)x_1''(t)+A_2'(t)x_2''(t)+A_3'(t)x_3''(t)=f(t).\end{cases}$$

We solve the system as functional-algebraic equations by the Cramer rule and found:

$$A_1(t)=\frac{1}{12c^2}\int_0^t \frac{1}{x_1(s)} f(s)ds, \tag{9}$$

$$A_2(t)=-\frac{1}{12c^2}\int_0^t \left[x_1(s)x_2(s)+\sqrt{3}x_1(s)x_3(s)\right] f(s)ds, \tag{10}$$

$$A_3(t)=\frac{1}{12c^2}\int_0^t \left[\sqrt{3}x_1(s)x_2(s)-x_1(s)x_3(s)\right] f\left(s\right)ds, \tag{11}$$

where $2c=\sqrt[3]{\lambda}$. Substituting (9)-(11) into (8) and taking into account (7), we obtain a particular solution of the equation (5) as

$$\tilde{x}(t,\lambda)=\frac{1}{\sqrt[3]{\lambda}}\int_0^t Q(t,s,\lambda) f(s)ds, \tag{12}$$

where

$$Q(t,s,\lambda)=\frac{1}{3}\left\{e^{-\sqrt[3]{\lambda}(t-s)}-2e^{\frac{\sqrt[3]{\lambda}}{2}(t-s)}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}(s-t)+\frac{\pi}{6}\right)\right\}.$$

The presentation (12) is a particular solution of the equation (1). So, the general solution of the equation (5) can be presented as:

$$x(t)=B_1x_1(t)+B_2x_2(t)+B_3x_3(t)+\frac{1}{\sqrt[3]{\lambda}}\int_0^t Q(t,s,\lambda) f(s)ds. \tag{13}$$

To find the unknown (arbitrary) coefficients B_k $(k=1,2,3)$, we use the boundary conditions (2). Then from the presentation (13) we obtain

$$\begin{cases} B_1x_1=\frac{1}{3}x_1\varphi_1-\frac{1}{6c}x_1\varphi_2+\frac{1}{12c^2}x_1\varphi_3, \\ B_2x_2=\frac{2}{3}x_2\varphi_1+\frac{1}{6c}x_2\varphi_2-\frac{1}{12c^2}x_2\varphi_3, \\ B_3x_3=\frac{1}{2\sqrt{3}c}x_3\varphi_2+\frac{1}{4\sqrt{3}c^2}x_3\varphi_3. \end{cases}$$

Hence, we have that

$$B_1x_1+B_2x_2+B_3x_3=\frac{x_1+2x_2}{3}\varphi_1+$$
$$+\frac{-x_1+x_2+\sqrt{3}x_3}{6c}\varphi_2+\frac{x_1-x_2+\sqrt{3}x_3}{12c^2}\varphi_3. \tag{14}$$

Substituting (14) into (13) and taking into account (3), we obtain

$$x(t,\lambda)=P(t,\lambda)+$$
$$+\frac{1}{\sqrt[3]{\lambda}}\left[\nu\sum_{i=1}^{p}\tau_i\int_0^t Q(t,s,\lambda)\alpha_i(s)ds+\int_0^t Q(t,s,\lambda)F\left(s,\int_0^T G(\theta)x(\theta,\lambda)d\theta\right)ds\right], \tag{15}$$

where

$$P(t,\lambda)=\varphi_1\psi_1(t,\lambda)+\frac{1}{\sqrt[3]{\lambda}}\varphi_2\psi_2(t,\lambda)+\frac{1}{\sqrt[3]{\lambda^2}}\varphi_3\psi_3(t), \tag{16}$$

$$Q(t,s,\lambda)=\frac{1}{3}\left[e^{-\sqrt[3]{\lambda}(t-s)}+2e^{\frac{\sqrt[3]{\lambda}}{2}(t-s)}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}(t-s)+\frac{\pi}{6}\right)\right], \tag{17}$$

$$\psi_1(t,\lambda)=\frac{1}{3}\left[e^{-\sqrt[3]{\lambda}t}+2e^{\frac{\sqrt[3]{\lambda}t}{2}}\cos\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}t\right], \tag{18}$$

$$\psi_2(t,\lambda)=\frac{1}{3}\left[e^{-\sqrt[3]{\lambda}t}-2e^{\frac{\sqrt[3]{\lambda}}{2}t}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}t+\frac{\pi}{6}\right)\right], \tag{19}$$

$$\psi_{3,n}(t,\lambda)=\frac{1}{3}\left[e^{-\sqrt[3]{\lambda}t}-2e^{\frac{\sqrt[3]{\lambda}}{2}t}\sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda}t-\frac{\pi}{6}\right)\right]. \tag{20}$$

There is another unknown quantity in (15). To find it we substitute (15) into (4), and obtain a system of algebraic equations (SAE)

$$\tau_i=\nu\sum_{j=1}^{p}\tau_j\Phi_{i,j}(\lambda)+\Psi_i(x,\lambda),\quad i=1,2,\cdots,p, \tag{21}$$

$$\Phi_{i,j}(\lambda)=\frac{1}{\sqrt[3]{\lambda}}\int_0^T\beta_i(s)\int_0^s Q(s,\theta,\lambda)\alpha_j(\theta)\,d\theta ds, \tag{22}$$

$$\Psi_i(x,\lambda)=\int_0^T\beta_i(s)P(s,\lambda)ds+\frac{1}{\sqrt[3]{\lambda}}\int_0^T\beta_i(s)\int_0^s Q(s,\theta,\lambda)\times$$

$$\times F\left(\theta,\int_0^T G(\xi)x(\xi,\lambda)d\xi\right)d\theta ds. \tag{23}$$

To solve the SAE we consider the following determinants

$$Z(\nu,\lambda)=\begin{vmatrix}1-\nu\Phi_{11} & \nu\Phi_{12} & \cdots & \nu\Phi_{1p}\\ \nu\Phi_{21} & 1-\nu\Phi_{22} & \cdots & \nu\Phi_{2p}\\ \cdots & \cdots & \cdots & \cdots\\ \nu\Phi_{p1} & \nu\Phi_{p2} & \cdots & 1-\nu\Phi_{pp}\end{vmatrix}, \tag{24}$$

$$Z(x,\nu,\lambda)=\begin{vmatrix}1-\nu\Phi_{11} & \ldots & \nu\Phi_{1(i-1)} & \Psi_1 & \nu\Phi_{1(i+1)} & \ldots & \nu\Phi_{1p}\\ \nu\Phi_{21} & \ldots & \nu\Phi_{2(i-1)} & \Psi_2 & \nu\Phi_{2(i+1)} & \ldots & \nu\Phi_{2p}\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ \nu\Phi_{p1} & \ldots & \nu\Phi_{p(i-1)} & \Psi_p & \nu\Phi_{p(i+1)} & \ldots & 1-\nu\Phi_{pp}\end{vmatrix}, \tag{25}$$

where $\Phi_{ij}=\Phi_{ij}(\lambda),\ \Psi_\kappa=\Psi_\kappa(x,\lambda),\ \kappa=\overline{1,p}$.

SAE (21) is uniquely solvable for any finite right-hand sides, if the following non-degeneracy condition for the Fredholm determinant is satisfied: $Z(\nu,\lambda)\neq 0.$ The determinant (24) $Z(\nu,\lambda)$ is a polynomial with respect to ν of degree not higher p. The equation $Z(\nu,\lambda)=0$ has at most p different real roots. We denote them by $\theta_\ell\,(\ell=\overline{1,p_\ell}\,,\ \ 1\leq p_\ell\leq p)$. Then $\nu=\nu_\ell=\theta_\ell$ called the irregular values of the parameter ν. Other values of the parameter $\nu\neq\theta_\ell$, for which $\left|Z(\nu,\lambda)\right|>0$ are called regular.

For regular values of the parameter ν the solution of the SAE (21) has the form

$$\tau_\kappa(\nu,\lambda)=\frac{Z_\kappa(x,\nu,\lambda)}{Z(\nu,\lambda)},\quad \kappa=\overline{1,p}, \tag{26}$$

where $Z_\kappa(\nu,\lambda)$ is defined from (25). Substituting for regular values of the parameter ν the presentation of solution (26) of the SAE (21) into representation (15), we derive a nonlinear system of functional integral equations (NSFIE)

$$x(t,\lambda)=P(t,\lambda)+\frac{\nu}{\sqrt[3]{\lambda}}\sum_{i=1}^{p}\frac{Z_\kappa(x,\nu,\lambda)}{Z(\nu,\lambda)}\int_0^t Q(t,s,\lambda)\alpha_i(s)\,ds+$$
$$+\frac{1}{\sqrt[3]{\lambda}}\int_0^t Q(t,s,\lambda)F\left(s,\int_0^T G(\theta)\,x(\theta,\lambda)d\theta\right)ds. \tag{27}$$

We note that the functions in (17)-(20) become zero at some values of parameter λ. We obtain the following transcendental equation

$$\sin\left(\frac{\sqrt{3}}{2}y+\frac{\pi}{6}\right)=-\frac{1}{2}e^{\frac{-3}{2}y},\quad y=\sqrt[3]{\lambda}\,(t-s)>0$$

for the case of function (17), and

$$\cos\frac{\sqrt{3}}{2}y=-\frac{1}{2}e^{\frac{-3}{2}y},\quad y=\sqrt[3]{\lambda}\,t>0$$

for the case of function (18), respectively. Functions in the formulas (19) and (20) become zero at some values of parameter λ. We replace these equations by the following transcendental equations

$$\sin\left(\frac{\sqrt{3}}{2}y+\frac{\pi}{6}\right)=\frac{1}{2}e^{\frac{-3y}{2}},\quad y=\sqrt[3]{\lambda}\,t>0,$$

$$\sin\left(\frac{\sqrt{3}}{2}y-\frac{\pi}{6}\right)=\frac{1}{2}e^{\frac{-3y}{2}},\quad y=\sqrt[3]{\lambda}\,t>0,$$

respectively.

The values of parameter λ, for which the functions (17)-(20) become zero, we denote by $\Lambda_j,\quad j=1,2,3,4$, respectively. However, from the fact $\Lambda_1\cap\Lambda_2\cap\Lambda_3\cap\Lambda_4=\varnothing$ we deduce that the problem (1)-(2) is correct.

Solvability of the NSFIE (27)

To prove the unique solvability of the mixed problem (1)-(2) we require in some properties of the given functions (17)-(20).

Theorem 1. Let the following conditions be fulfilled:

$$\text{1).}\ \alpha_0 \sum_{\kappa=1}^{p} \left| \frac{Z_\kappa(x^0,\nu,\lambda)}{Z(\nu,\lambda)} \right| \le \delta_0, \ \ \alpha_0 = \max_{0\le t\le T} \int_0^t \alpha_j(s)\,ds, \ \ 0<\delta_0, \alpha_0 = const < \infty;$$

$$\text{2).}\ \| F(t,\cdot) \| \le \delta_1, \ \ 0<\delta_1 = const < \infty;$$

$$\text{3).}\ | F(t,u_1) - F(t,u_2) | \le l_0 | u_1 - u_2 |, \ \ 0 < l_0 = const < \infty;$$

4). $\rho < 1$, where ρ determines from (32) below.

Then for regular values of the parameter ν NSFIE (27) has a unique solution in the space $C[0,T]$ of continuous functions.

Proof. We define the successive approximations for NSFIE (27) as:

$$\begin{cases} x^0(t,\nu,\lambda) = P(t,\nu,\lambda), \\ x^{m+1}(t,\nu,\lambda) = J(t;x^m), \ m = 0,1,2,3,... \end{cases} \tag{28}$$

We estimate the zero approximation. By virtue of formulas (17)-(20),
we can put

$$\max\left\{ \max_t | Q(t,s,\lambda) |; \max_{k=1,2,3} \max_t | \psi_k(t,\lambda) | \right\} \le M_0 < \infty, \ \ 0 < M_0 = const < \infty.$$

Taking into account (16), for the first approximation, from (28) we have

$$\| x^0(t,\nu,\lambda) \|_{C[0,T]} \le \max_{0\le t\le T} | P(t,\lambda) | \le M_0 \left[|\varphi_1| + \frac{1}{\sqrt[3]{\lambda}} |\varphi_2| + \frac{1}{\sqrt[3]{\lambda^2}} |\varphi_3| \right]. \tag{29}$$

Due to the conditions of the Theorem 1, for the first difference of consecutive approximation functions $x^1(t) - x^0(t)$ we obtain

$$\| x^1(t,\nu,\lambda) - x^0(t,\nu,\lambda) \|_{C[0,T]} \le \frac{|\nu|}{\sqrt[3]{\lambda}} \sum_{\kappa=1}^{p} \left| \frac{Z_\kappa(x^0,\nu,\lambda)}{Z(\nu,\lambda)} \right| \int_0^t | Q(t,s,\lambda) | \alpha_\kappa(s)\,ds +$$

$$+ \frac{1}{\sqrt[3]{\lambda}} \int_0^t | Q(t,s,\lambda) | \left| F\left(s, \int_0^T G(\theta) x^0(\theta,\lambda)\,d\theta \right) \right| ds \le \frac{M_0 \delta_0}{\sqrt[3]{\lambda}} \left(|\nu| \alpha_0 \delta_0 + \delta_1 \right). \tag{30}$$

Now we consider the arbitrary consecutive difference $x^{m+1}(t) - x^m(t)$. Taking into account the formulas (22) and (23), we obtain the following estimate

$$\| x^{m+1}(t,\nu,\lambda) - x^m(t,\nu,\lambda) \|_{C[0,T]} \le$$

$$\le \frac{|\nu|}{\sqrt[3]{\lambda}} \sum_{i=1}^{p} \left| \frac{Z_\kappa(x^m,\nu,\lambda) - Z_\kappa(x^{m-1},\nu,\lambda)}{Z(\nu,\lambda)} \right| \int_0^t | Q(t,s,\lambda) | \alpha_i(s)\,ds +$$

$$+\frac{l_0}{\sqrt[3]{\lambda}}\int_0^t |Q(t,s,\lambda)| \left| \int_0^T G(\theta) \left| x^m(\theta,\nu,\lambda) - x^{m-1}(\theta,\nu,\lambda) \right| d\theta \right| ds \leq$$

$$\leq \rho \cdot \left\| x^{m+1}(t,\nu,\lambda) - x^m(t,\nu,\lambda) \right\|_{C[0,T]}, \qquad (31)$$

where

$$\rho = \frac{M_0}{\sqrt[3]{\lambda}}\left(|\nu| \bar{\delta}_0 \alpha_0 + l_0 G_0 \right), \quad \bar{\delta}_0 = \sum_{\kappa=1}^{p} \left| \frac{\bar{Z}_\kappa(\nu,\lambda)}{Z(\nu,\lambda)} \right|, \quad 0 < \int_0^T G(s)\, ds = G_0 < \infty, \quad (32)$$

$$\bar{Z}_\kappa(\nu,\lambda) = \begin{vmatrix} 1-\nu\Phi_{11} & \dots & \nu\Phi_{1(\kappa-1)} & 1 & \nu\Phi_{1(\kappa+1)} & \dots & \nu\Phi_{1p} \\ \nu\Phi_{21} & \dots & \nu\Phi_{2(\kappa-1)} & 1 & \nu\Phi_{2(\kappa+1)} & \dots & \nu\Phi_{2p} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \nu\Phi_{p1} & \dots & \nu\Phi_{p(\kappa-1)} & 1 & \nu\Phi_{p(\kappa+1)} & \dots & 1-\nu\Phi_{pp} \end{vmatrix}.$$

From the estimates (29)-(31) it follows that the operator $J(t;x)$ on the right-hand side of (27) is contracting and there is unique fixed point. So, the existence and uniqueness of the solution $x(t) \in C[0,T]$ to NSFIE (27) are proved. The theorem is proved.

Continuously dependence of the solution to NSFIE from parameter λ

In this section we use the following obvious lemma.

Lemma. For two values λ_1, λ_2 of positive parameter λ there true the following estimates

$$\left| e^{-\sqrt[3]{\lambda_1}(t-s)} - e^{-\sqrt[3]{\lambda_2}(t-s)} \right| \leq L_{01} |\lambda_1 - \lambda_2|, \quad 0 < L_{01} = const;$$

$$\left| \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda_1}(t-s) + \frac{\pi}{6} \right) - \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda_2}(t-s) + \frac{\pi}{6} \right) \right| \leq L_{02} |\lambda_1 - \lambda_2|, \quad 0 < L_{02} = const;$$

$$\left| e^{\sqrt[3]{\lambda_1}\frac{t-s}{2}} - e^{\sqrt[3]{\lambda_2}\frac{t-s}{2}} \right| \leq L_{03} |\lambda_1 - \lambda_2|, \quad 0 < L_{03} = const;$$

$$\left| \cos\frac{\sqrt{3}}{2}\sqrt[3]{\lambda_1}(t-s) - \cos\frac{\sqrt{3}}{2}\sqrt[3]{\lambda_2}(t-s) \right| \leq L_{02} |\lambda_1 - \lambda_2|, \quad 0 < L_{02} = const;$$

$$\left| \frac{1}{\sqrt[3]{\lambda_1}} - \frac{1}{\sqrt[3]{\lambda_2}} \right| \leq L_{04} |\lambda_1 - \lambda_2|, \quad L_{04} = const;$$

$$\left| \frac{1}{\sqrt[3]{\lambda_1^2}} - \frac{1}{\sqrt[3]{\lambda_2^2}} \right| \leq L_{05} |\lambda_1 - \lambda_2|, \quad L_{05} = const.$$

Theorem 2. Let be fulfilled the conditions of the Theorem 1. Then the following estimate

$$\left\| x(t,\lambda_1) - x(t,\lambda_2) \right\|_{C[0,T]} \leq L_M |\lambda_1 - \lambda_2|, \quad 0 < L_M = const \qquad (33)$$

holds.

Proof. By virtue of the Lemma for the function (17) we obtain

$$\left| Q(t,s,\lambda_1) - Q(t,s,\lambda_2) \right| \le \frac{1}{3}\left| e^{-\sqrt[3]{\lambda_1}(t-s)} - e^{-\sqrt[3]{\lambda_2}(t-s)} \right| +$$

$$+\frac{2}{3}\left| e^{\frac{\sqrt[3]{\lambda_1}}{2}(t-s)} - e^{\frac{\sqrt[3]{\lambda_2}}{2}(t-s)} \right| \cdot \left| \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda_1}(t-s) + \frac{\pi}{6} \right) \right| + \frac{2}{3} e^{\frac{\sqrt[3]{\lambda_2}}{2}(t-s)} \times$$

$$\times \left| \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda_1}(t-s) + \frac{\pi}{6} \right) - \sin\left(\frac{\sqrt{3}}{2}\sqrt[3]{\lambda_2}(t-s) + \frac{\pi}{6} \right) \right| \le L_1 \left| \lambda_1 - \lambda_2 \right|, \quad (34)$$

where

$$L_1 = L_{01} + L_{03} + e^{\sqrt[3]{\lambda_2}T} L_{02}.$$

By virtue of Lemma, for the function (18) we obtain

$$\left| \psi_1(t,\lambda_1) - \psi_1(t,\lambda_2) \right| \le \frac{1}{3}\left| e^{-\sqrt[3]{\lambda_1}t} - e^{-\sqrt[3]{\lambda_2}t} \right| + \frac{2}{3}\left| e^{\frac{\sqrt[3]{\lambda_1}}{2}t} - e^{\frac{\sqrt[3]{\lambda_2}}{2}t} \right| \cdot \left| \cos\frac{\sqrt{3}}{2}\sqrt[3]{\lambda_1}t \right| +$$

$$+\frac{2}{3} e^{\frac{\sqrt[3]{\lambda_2}}{2}t} \left| \cos\frac{\sqrt{3}}{2}\sqrt[3]{\lambda_1}t - \cos\frac{\sqrt{3}}{2}\sqrt[3]{\lambda_2}t \right| \le L_1 \left| \lambda_1 - \lambda_2 \right|. \quad (35)$$

By similarly way for the functions (19) and (20) we obtain

$$\left| \psi_j(t,\lambda_1) - \psi_j(t,\lambda_2) \right| \le L_1 \left| \lambda_1 - \lambda_2 \right|, \quad j = 2,3. \quad (36)$$

By the aid of the estimates (34)-(36), taking properties of the functions (22), (23) and matrix (25), for the NSFIE (27) we derive

$$\left\| x(t,\nu,\lambda_1) - x(t,\nu,\lambda_2) \right\|_{C[0,T]} \le \max_{0\le t\le T} \left| P(t,\lambda_1) - P(t,\lambda_2) \right| +$$

$$+\left|\nu\right| \left| \frac{1}{\sqrt[3]{\lambda_1}} - \frac{1}{\sqrt[3]{\lambda_2}} \right| \left| \sum_{\kappa=1}^{p} \right| \frac{Z_\kappa(x,\nu,\lambda_1)}{Z(\nu,\lambda_1)} \left| \max_{0\le t\le T} \int_0^t \left| Q(t,s,\lambda_1) \right| \alpha_j(s)\, ds + \right.$$

$$+\frac{\left|\nu\right|}{\sqrt[3]{\lambda_2}} \sum_{\kappa=1}^{p} \left| \frac{Z_\kappa(x,\nu,\lambda_1)}{Z(\nu,\lambda_1)} \right| \max_{0\le t\le T} \int_0^t \left| Q(t,s,\lambda_1) - Q(t,s,\lambda_2) \right| \alpha_j(s)\, ds +$$

$$+\alpha_0 M_0 \frac{\left|\nu\right|}{\sqrt[3]{\lambda_2}} \sum_{\kappa=1}^{p} \left| Z_\kappa(x,\nu,\lambda_1) \right| \left| \frac{1}{Z(\nu,\lambda_1)} - \frac{1}{Z(\nu,\lambda_2)} \right| +$$

$$+\alpha_0 M_0 \frac{\left|\nu\right|}{\sqrt[3]{\lambda_2}} \left| \frac{1}{Z(\nu,\lambda_2)} \right| \sum_{\kappa=1}^{p} \left| Z_\kappa(x,\nu,\lambda_1) - Z_\kappa(x,\nu,\lambda_2) \right| +$$

$$+\left| \frac{1}{\sqrt[3]{\lambda_1}} - \frac{1}{\sqrt[3]{\lambda_2}} \right| \max_{0\le t\le T} \int_0^t Q(t,s,\lambda_1) \left| F(s,\cdot) \right| ds +$$

$$+\frac{1}{\sqrt[3]{\lambda_2}}\max_{0\le t\le T}\int_0^t \left|Q(t,s,\lambda_1)-Q(t,s,\lambda_2)\right|\left|F(s,\cdot)\right|ds+$$

$$+\frac{M_0}{\sqrt[3]{\lambda_2}}l_0 p\int_0^T \left|G(t)\right|\left|x(t,\lambda_1)-x(t,\lambda_2)\right|dt. \quad (37)$$

By the aid of Lemma, estimates (34)-(37) and conditions of the Theorem 1, we obtain

$$\| x(t,\nu,\lambda_1)-x(t,\nu,\lambda_2)\|_{C[0,T]}\le M_0\left[L_1|\varphi_1|+(L_1+L_{04})|\varphi_2|+(L_1+L_{05})|\varphi_3|\right]|\lambda_1-\lambda_2|+$$

$$+|\nu|M_0\delta_0 L_{04}|\lambda_1-\lambda_2|+\frac{|\nu|}{\sqrt[3]{\lambda_2}}M_0\delta_0 L_1|\lambda_1-\lambda_2|+$$

$$+\delta_0 M_0^2\frac{|\nu|}{\sqrt[3]{\lambda_2}}\left[L_{04}+L_1\frac{1}{\sqrt[3]{\lambda_2}}\right]|\lambda_1-\lambda_2|+\alpha_0\beta_0\delta_2 M_0^2\frac{|\nu|}{\sqrt[3]{\lambda_2^2}}\int_0^T G(t)\left|x^m(t,\lambda_1)-x^{m-1}(t,\lambda_2)\right|dt+$$

$$+M_0 L_{04}T\left|F(t,\cdot)\right||\lambda_1-\lambda_2|+L_1\frac{T}{\sqrt[3]{\lambda_2}}\left|F(t,\cdot)\right||\lambda_1-\lambda_2|+$$

$$+\frac{l_0 M_0}{\sqrt[3]{\lambda_2}}\int_0^T G(t)\left|x(t,\lambda_1)-x(t,\lambda_2)\right|dt, \quad (38)$$

where

$$\beta_0=\sum_{\kappa=1}^{p}\left|\bar{\bar{Z}}(\nu)\right|\int_0^T \beta_\kappa(t)dt,$$

$$\bar{\bar{Z}}(\nu)=\begin{vmatrix} 1-\nu\Delta_{11} & \nu\Delta_{12} & \cdots & \nu\Delta_{1p} \\ \nu\Delta_{21} & 1-\nu\Delta_{22} & \cdots & \nu\Delta_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ \nu\Delta_{p1} & \nu\Delta_{p2} & \cdots & 1-\nu\Delta_{pp} \end{vmatrix}, \quad \Delta_{ij}=\int_0^T\beta_i(s)\int_0^s\alpha_j(\theta)\,d\theta ds.$$

It is not difficult to check that from (38) we obtain

$$\| x(t,\nu,\lambda_1)-x(t,\nu,\lambda_2)\|_{C[0,T]}\le M_4|\lambda_1-\lambda_2|+\rho\cdot\| x(t,\nu,\lambda_1)-x(t,\nu,\lambda_2)\|_{C[0,T]}, \quad (39)$$

where

$$M_4=M_0\left[L_1|\varphi_1|+(L_1+L_{04})|\varphi_2|+(L_1+L_{05})|\varphi_3|\right]+$$

$$+|\nu|M_0\delta_0 L_{04}+\frac{|\nu|}{\sqrt[3]{\lambda_2}}M_0\delta_0 L_1+\delta_0 M_0^2\frac{|\nu|}{\sqrt[3]{\lambda_2}}\left[L_{04}+L_1\frac{1}{\sqrt[3]{\lambda_2}}\right]+$$

$$+M_0 L_{04}T\left|F(t,\cdot)\right|+L_1\frac{T}{\sqrt[3]{\lambda_2}}\left|F(t,\cdot)\right|.$$

From the estimate (39) we obtain (33). Theorem 2 is proved.

Conclusion

It is considered a third order nonlinear Fredholm integro-differential equation (1) with initial value conditions (2) and with two real parameters ν, ω. A nonlinear Volterra-Fredholm functional integral equation (27) is derived. Theorem on a uniqueness and existence of the solution of initial value problem (1), (2) is proved for regular values of parameter ν. The method of compressing mapping is applied for the equation (27) in Banach space $C[0,T]$ of continuous functions. For the solution of the problem (1), (2) is studied continuous dependence on parameter λ.

We hope that this work can serve as a basis for further development of the theory of partial differential and integro-differential equations of the third and higher orders.

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