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BLOW-UP OF SMOOTH SOLUTIONS OF THE PROBLEM FOR THE KORTEWEG-DE VRIES-BURGERS EQUATION WITH THE HILFER FRACTIONAL DIFFERENTIAL OPERATOR

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Abstract. *This work is devoted to studying the non-existence of the global-in-time solutions for the Korteweg-de Vries-Burgers equation including Hilfer time fractional differential operator which in particular cases of the parameters follows the classical and other time-fractional Korteweg-de Vries-Burgers equation. Applying the method of nonlinear capacity which was suggested by S.I. Pokhozhaev for some initial-boundary value problems, it has been obtained sufficient conditions for the non-existence of global solutions.*

Keywords: *Hilfer derivative, the method of nonlinear capacity, non-existence of the solution.*

РАЗРУШЕНИЕ ГЛАДКИХ РЕШЕНИЙ ЗАДАЧИ ДЛЯ УРАВНЕНИЯ КОРТЕВЕГА-ДЕ ВРИСА-БЮРГЕРСА С ОПЕРАТОРОМ ДРОБНОГО ДИФФЕРЕНЦИАЛА ХИЛЬФЕРА

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Аннотация. *Настоящая работа посвящена изучению отсутствия глобальных по времени решений уравнения Кортевега-де Фриза-Бюргерса, включающего дробно-дифференциальный оператор Гильфера по времени, который в частных случаях параметров следует классическому и другим дробным по времени уравнениям Кортевега-Бюргерса. уравнение де Фриза-Бюргерса. Применяя метод нелинейной емкости, предложенный С.И. Похожаевым для некоторых начально-краевых задач, получены достаточные условия отсутствия глобальных решений.*

Ключевые слова: *производная Гильфера, метод нелинейной емкости, отсутствие решения.*

1. PRELIMINARIES. In this section, we give some basic concepts of fractional calculus.

Definition 1.1. [1] Let $f \in L([a,b])$. The following integrals

$$I_{a+}^{\alpha}[f](t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)ds \tag{1.1}$$

and

$$I_{b-}^{\alpha}[f](t)=\frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1} f(s)ds \tag{1.2}$$

are called the left-sided and the right-sided Riemann-Liouville integrals of the fractional order $\alpha > 0$, respectively, where $\Gamma(z)$ denotes the Euler's gamma function.

Definition 1.2. The Riemann-Liouville left-sided fractional derivative $D_{a+}^{\alpha} f$ of order $\alpha (0 < \alpha < 1)$ is defined by

$$D_{a+}^{\alpha}[f](t) = \frac{d}{dt} I_{a+}^{1-\alpha}[f](t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s) ds. \tag{1.3}$$

Definition 1.3. The Riemann-Liouville right-sided fractional derivative $D_{b-}^{\alpha} f$ of order $\alpha (0 < \alpha < 1)$ is defined by

$$D_{b-}^{\alpha}[f](t) = -\frac{d}{dt} I_{b-}^{1-\alpha}[f](t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (s-t)^{-\alpha} f(s) ds. \tag{1.4}$$

Definition 1.4. The Hilfer derivative $D_{a+}^{\alpha,\beta} f$ of order $0 < \alpha < 1$ and type $0 \le \beta \le 1$ is defined by

$$D_{a+}^{\alpha,\beta}[f](t) = I_{a+}^{\beta(1-\alpha)} \frac{d}{dt} I_{a+}^{(1-\beta)(1-\alpha)}[f](t) \tag{1.5}$$

where I_{a+}^{σ}, $\sigma > 0$ is the Riemann-Liouville fractional integral.

The Hilfer derivative was introduced in [2], [3]. These references provide information about the applications of this derivative and how it arises. It is easy to see that this derivative interpolates the Riemann-Liouville fractional derivative $(\beta = 0)$ and the Caputo fractional derivative $(\beta = 1)$ (see [1]).

The fractional integration by parts is defined as follows.

Lemma 1.1. Le $\alpha > 0, p \ge 1, q \ge 1$ and $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$ ($p \ne 1$ and $q \ne 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). If $\varphi \in L_p(a,b)$ and $\psi \in L_q(a,b)$, then

$$\int_a^b \varphi(t) I_{a+}^{\alpha}[\psi](t) dt = \int_a^b \psi(t) I_{b-}^{a}[\varphi] dt. \tag{1.6}$$

2. NON-EXISTENCE OF THE SOLUTION OF TIME-FRACTIONAL KORTEWEG-DE-VRIES-BURGERS EQUATION

Let denote by $\Pi_{a,b}$ a rectangular domain of $\mathbf{R}^2$, i.e $\Pi_{a,b} = \{(t,x) \in \mathbf{R}^2 : 0 < t < T, a < x < b\}$. In the domain $\Pi_{a,b}$, we consider the time-fractional Korteweg-de Vries-Burgers equation

$$D_{0+,t}^{\alpha,\beta} u(t,x) + u(t,x) u_x(t,x) + u_{xxx}(t,x) = \nu u_{xx}(t,x) \tag{2.1}$$

with the following initial condition

$$I_{0+,t}^{\gamma-1} u(0,x) = u_0(x),\ x \in [a,b], \tag{2.2}$$

where $D_{a+}^{\alpha,\beta}$ is the Hilfer derivative of order $0 < \alpha < 1$ and type $0 \le \beta \le 1$ with respect to t, $\nu > 0$ and $u_0(x)$ is a given function.

If $\beta = 1$ then the equation (2.1) takes the form which studied in [4]. And when $\beta = 1$ and $\alpha = 1$ it becomes the classical Korteweg-de Vries-Burgers equation [5]. We should note the

Korteweg-de Vries-Burgers equation can be applied as the mathematical model for many real-life processes [5].

Our aim is to investigate blow-up solutions of the problem (2.1)-(2.2). To do this we apply the method of nonlinear capacity. This concept for analyzing blow-up of solutions nonlinear equations was suggested by Pokhozhaev in [6].

We consider a class $\Phi\left(\Pi_{a,b}\right)$ of test functions $\varphi\left(t,x\right)$, defined on the domain $\Pi_{a,b}$ with arbitrary parameters $T>0$, $a,b\in\mathbf{R}$, have the following properties:

(i) $\varphi_t,\varphi_{xx},\varphi_{xxx}\in C\left(\Pi_{a,b}\right)$;

(ii) $\varphi_x\geq 0$ in $\Pi_{a,b}$;

(iii) $I_{T-,t}^{\beta(1-\alpha)}\varphi\left(x,t\right)=0$ at $t=T$ and $x\in\left(a,b\right)$;

(iv) $\zeta\left(\Pi_{a,b}\right)=\iint\limits_{\Pi_{a,b}}\frac{\left(L^*\varphi\right)^2}{\varphi_x}dtdx<+\infty$,

where $L^*\varphi=-I_{T-,t}^{(1-\beta)(1-\alpha)}D_{T-,t}^{1-\beta(1-\alpha)}\varphi+\nu\varphi_{xx}-\varphi_{xxx}$.

Suppose that there exists an $T>0$ for which weak solution of the problem (2.1)-(2.2) satisfying $u_{xx},D_{0+,t}^{\alpha,\beta}u\in C\left(\left[a,b\right]\times\left[0,t\right]\right)$.

By multiplying the equation (2.1) by a test function $\varphi\in\Phi\left(\Pi_{a,b}\right)$ and then integrating over $\Pi_{a,b}$ obtained equality, we get

$$\iint\limits_{\Pi_{a,b}}\varphi\left(t,x\right)D_{0+,t}^{\alpha,\beta}u\left(t,x\right)dtdx+\iint\limits_{\Pi_{a,b}}\varphi\left(t,x\right)u\left(t,x\right)u_x\left(t,x\right)dtdx+$$
$$+\iint\limits_{\Pi_{a,b}}\varphi\left(t,x\right)u_{xxx}\left(t,x\right)dtdx=\nu\iint\limits_{\Pi_{a,b}}\varphi\left(t,x\right)u_{xx}\left(t,x\right)dtdx\cdot \tag{2.3}$$

Applying the rule of integration by parts, it is easy to obtain the following equalities

$$\iint\limits_{\Pi_{a,b}}\varphi\left(t,x\right)u\left(t,x\right)u_x\left(t,x\right)dtdx=$$
$$=\frac{1}{2}\int_0^T u^2\left(t,x\right)\varphi\left(t,x\right)|_a^b dt-\frac{1}{2}\iint\limits_{\Pi_{a,b}}u^2\left(t,x\right)\varphi_x\left(t,x\right)dtdx, \tag{2.4}$$

$$\iint\limits_{\Pi_{a,b}}\varphi\left(t,x\right)u_{xx}\left(t,x\right)dtdx=$$
$$=\int_0^T\left[u_x\left(t,x\right)\varphi\left(t,x\right)-u\left(t,x\right)\varphi_x\left(t,x\right)\right]\Big|_a^b dt-\iint\limits_{\Pi_{a,b}}u\left(t,x\right)\varphi_{xx}\left(t,x\right)dtdx, \tag{2.5}$$

$$\iint\limits_{\Pi_{a,b}}\varphi\left(t,x\right)u_{xxx}\left(t,x\right)dtdx=$$
$$\int_0^T\left[\varphi u_{xx}-\varphi_x u_x+\varphi_{xx}u\right]\Big|_a^b dt-\iint\limits_{\Pi_{a,b}}\varphi_{xxx}\left(t,x\right)u\left(t,x\right)dtdx \quad . \tag{2.6}$$

Using Definition 1.4 and applying Lemma 1.1, we have

$$\iint\limits_{\Pi_{a,b}}\varphi\left(t,x\right)D_{0+,t}^{\alpha,\beta}u\left(t,x\right)dtdx=\iint\limits_{\Pi_{a,b}}\varphi\left(t,x\right)I_{0+}^{\beta(1-\alpha)}\frac{d}{dt}I_{0+}^{(1-\beta)(1-\alpha)}u\left(t,x\right)dtdx=$$
$$=\iint\limits_{\Pi_{a,b}}I_{T-,t}^{\beta(1-\alpha)}\varphi\left(t,x\right)\frac{d}{dt}I_{0+}^{(1-\beta)(1-\alpha)}u\left(t,x\right)dtdx.$$

Hence, applying the rule of integration by parts and using Lemma 2.1, we obtain

$$\iint_{\Pi_{a,b}} \varphi(t,x) D_{0+,t}^{\alpha,\beta} u(t,x)\, dtdx = \int_a^b \left\{ I_{0+,t}^{(1-\beta)(1-\alpha)} u(t,x) I_{T-,t}^{\beta(1-\alpha)} \varphi(t,x) \right\} \Big|_0^T dx -$$

$$- \iint_{\Pi_{a,b}} \frac{d}{dt} I_{T-,t}^{\beta(1-\alpha)} \varphi(t,x) I_{0+}^{(1-\beta)(1-\alpha)} u(t,x)\, dtdx =$$

$$\int_a^b \left\{ I_{0+,t}^{(1-\beta)(1-\alpha)} u(t,x) I_{T-,t}^{\beta(1-\alpha)} \varphi(t,x) \right\} \Big|_0^T dx - \iint_{\Pi_{a,b}} u(t,x) I_{T-,t}^{(1-\beta)(1-\alpha)} \frac{d}{dt} I_{T-,t}^{\beta(1-\alpha)} \varphi(t,x)\, dtdx .$$

Taking this and equalizes (2.4), (2.5) into account and also using Definition 2.3, from (2.3) we drive

$$\frac{1}{2} \iint_{\Pi_{a,b}} u^2(t,x) \varphi_x(t,x)\, dtdx = - \iint_{\Pi_{a,b}} u(t,x) \left(L^* \varphi \right)(t,x) +$$

$$+ \int_a^b \left\{ I_{0+,t}^{(1-\beta)(1-\alpha)} u(t,x) I_{T-,t}^{\beta(1-\alpha)} \varphi(t,x) \right\} \Big|_0^T dx + \int_0^T B\big(u(t,x), \varphi(t,x)\big) \Big|_a^b dt , \qquad (2.7)$$

where

$$B\big(u(t,x),\varphi(t,x)\big) = \frac{1}{2} u^2(t,x)\varphi(t,x) - \nu u_x(t,x)\varphi(t,x) + \nu u(t,x)\varphi_x(t,x) +$$

$$\varphi(t,x) u_{xx}(t,x) - \varphi_x(t,x) u_x(t,x) + \varphi_{xx}(t,x) u(t,x) .$$

Taking (2.2) and *(iii)* property of test functions, from the last we get

$$\frac{1}{2} \iint_{\Pi_{a,b}} u^2(t,x) \varphi_x(t,x)\, dtdx = - \iint_{\Pi_{a,b}} u(t,x) \left(L^* \varphi \right)(t,x)\, dtdx + \int_0^T B\big(u(t,x), \varphi(t,x)\big) \Big|_a^b dt -$$

$$- \int_a^b u_0(x) I_{T-,t}^{\beta(1-\alpha)} \varphi(x,t)\, dx . \qquad (2.8)$$

By applying Hőlder and Young's inequalities, it is easy to see that

$$\left| \iint_{\Pi_{a,b}} u(t,x) \left(L^* \varphi \right)(t,x)\, dtdx \right| = \left| \iint_{\Pi_{a,b}} u(t,x) \sqrt{\varphi_x(x,t)} \frac{\left(L^* \varphi \right)(t,x)}{\sqrt{\varphi_x(x,t)}}\, dtdx \right| \le$$

$$\left(\iint_{\Pi_{a,b}} u^2(t,x) \varphi_x(t,x)\, dtdx \right)^{1/2} \left(\iint_{\Pi_{a,b}} \frac{\left(\left(L^* \varphi \right)(t,x) \right)^2}{\varphi_x(x,t)}\, dtdx \right)^{1/2} \le$$

$$\frac{1}{2} \iint_{\Pi_{a,b}} u^2(t,x) \varphi_x(t,x)\, dtdx + \frac{1}{2} \iint_{\Pi_{a,b}} \frac{\left(\left(L^* \varphi \right)(t,x) \right)^2}{\varphi_x(x,t)}\, dtdx .$$

Taking this inequality and *(iv)* property of test functions into account from (2.8), we have

$$0 \le \frac{1}{2} \zeta\left(\Pi_{a,b} \right) + \int_0^T B\big(u(t,x), \varphi(t,x)\big) \Big|_a^b dt - \int_a^b u_0(x) I_{T-,t}^{\beta(1-\alpha)} \varphi(x,t) \Big|_{t=0} dx . \qquad (2.9)$$

The following theorem is valid:

Theorem 3.1. Suppose that the boundary conditions and the initial function $u_0(x) \in L[a,b]$ satisfy the following assumption: there exists a function $\varphi(x,t) \in \Phi\left(\Pi_{a,b} \right)$ such that $B\big(u(t,x), \varphi(t,x)\big) \Big|_a^b \in L[0,T]$ and the following inequality

$$\frac{1}{2} \zeta\left(\Pi_{a,b} \right) + \int_0^T B\big(u(t,x), \varphi(t,x)\big) \Big|_a^b dt - \int_a^b u_0(x) I_{T-,t}^{\beta(1-\alpha)} \varphi(x,t) \Big|_{t=0} dx < 0 . \qquad (2.10)$$

Then problem (2.1)-(2.2) does not admit a global-in-time solution in $\Pi_{a,b}$ with these initial and boundary conditions.

Proof. Let us assume the opposite i.e. the problem (2.1)-(2.2) admits a global-in-time solution in $\Pi_{a,b}$. Then we arrived at contradiction by virtue of inequalities (2.9) and (2.10).

Now, we consider the fractional Korteweg-de Vries-Burgers equation (2.1) with $\nu = 1$ in the rectangular domain $\Pi_{a,b} = \{(t,x) \in \mathbf{R}^2 : 0 < t < T, 0 < x < 1\}$ with the initial condition (2.2) and the following boundary conditions

$$u(t,0) = \tau_1(t),\ u_x(t,0) = \tau_2(t),\ 0 < t < T, \tag{2.11}$$

where τ_1 and τ_2 are given functions such that $\tau_1, \tau_2 \in L[0,T]$.

Multiply the time-fractional Korteweg-de Vries-Burgers equation (2.1) by a test function $\varphi \in \Phi(\Pi_{a,b})$, after some calculations and simplifications we obtain

$$0 < \frac{1}{2}\zeta(\Pi_{0,1}) + \int_0^T B(u(t,x),\varphi(t,x))\Big|_0^1 dt - \int_0^1 u_0(x) I_{T-,t}^{\beta(1-\alpha)} \varphi(x,t)\big|_{t=0} dx.$$

We take a test function satisfying the following boundary conditions:

$$\varphi(t,1) = 0,\ \varphi_x(t,1) = 0,\ 0 < t < T. \tag{2.12}$$

Then,

$$B(u,\varphi)\Big|_0^1 = -\left[\frac{1}{2}\tau_1^2(t) - \tau_2(t)\right]\varphi(t,0) - \tau_1(t)\varphi_x(t,0).$$

In this case, the following theorem is valid:

Theorem 2.2. Let the initial-boundary problem (2.1), (2.2), (2.11) be such that there exists a test function $\varphi \in \Phi(\Pi_{0,1})$ satisfying the boundary conditions (2.12) and also the following inequality

$$\frac{1}{2}\zeta(\Pi_{0,1}) < \int_0^T \left[\frac{1}{2}\tau_1^2(t)\varphi(t,0) - \tau_2(t)\varphi(t,0) + \tau_1(t)\varphi_x(t,0)\right] dt +$$

$$+ \int_0^1 u_0(x) I_{T-,t}^{\beta(1-\alpha)} \varphi(x,t)\big|_{t=0} dx. \tag{2.13}$$

Then the problem (2.1), (2.2), (2.11) does not admit a global-in-time solution in $\Pi_{0,1}$.

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