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ON COMPACTNESS TYPE EXTENSIONS OF TOPOLOGYCAL AND UNIFORM SPACES

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Abstract. *In this article extensions of real-complete Tychonoff and uniform spaces are considered, as well as locally compact paracompact and locally compact Lindelöff extensions of Tychonoff and uniform spaces.*
Keywords: *Uniform real complete extension, locally compact paracompact extension, locally compact Lindelöff extension.*

О РАСШИРЕНИЯХ ТИПА КОМПАКТНОСТИ ТОПОЛОГИЧЕСКИХ И РАВНОМЕРНЫХ ПРОСТРАНСТВ

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Абстракт. *В этой статье рассматриваются расширения вещественно полных тихоновских и равномерных пространств, а также локально компактно паракомпактные и локально компактно линделёфовые расширения тихоновских и равномерных пространств.*
Ключевые слова: *Равномерно вещественно полное расширение, локально компактный паракомпакт, локально компактное линделёфово расширение.*

ТОПОЛОГИЯЛЫК ЖАНА БИР КАЛЫПТУУ МЕЙКИНДИКТЕРДИН КОМПАКТУУЛУК ТҮРҮНҮН КЕНЕЙТҮҮСҮ

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Абстракт. *Бул макалада чыныгы толук тихоновдук жана бир калыптагы мейкиндиктерди кеңейтүүлөр, ошондой эле тихоновдук жана бир калыптагы мейкиндиктердин жергиликтүү компакттуу паракомпакттуу жана жергиликтүү компакттуу линделөфтүк кеңейтүүлөр каралат.*

Урунттуу сөздөр. *Бир калыптагы чыныгы толук кеңейтүү, жергиликтүү компакттуу паракомпакт, жергиликтүү компакттуу линделөфтүк кеңейтүү.*

The maximal real complete Tychonoff spaces extensions called the Hewitt extension. The real complete spaces introduced by Edwin Hewitt [1]. Uniform analogues analysis of other important classes topological spaces and formation all extensions of such Tychonoff spaces classes considered ([2],[3]).

Real complete extensions are considered in [4].

Let $C_U(X)$- the set of all uniformly continuous functions $f:(X,U) \to (R,E_R)$ and E_R-natural uniformity of the number line R.

Definition 1. A uniform space *(X, U)* is called a uniformly functional space, and the uniformity U is functional if the uniformity U is generated by some family of functions $C_U(X)$, i.e. U is generated by a family of coverings of the form $(f^{-1}\alpha: f\epsilon C_U(X), \alpha\epsilon E_R)$.

Proposition 1. For every uniformity of U on X existence, uniformity of U_F on X such that U_F is the maximum functional uniformity contained in the uniformity of U.

Definition 2. A uniform space (X, U) is called uniformly real complete if it is uniformly functionally and complete.

Theorem 1. Let (X, U) be a uniformly function space. Then its completion ($\widetilde{X}$, $\widetilde{U}$) is uniformly real complete, and its topological space (X,τ_U) will be real complete spaces.

Let (X, U) be an arbitrary uniform space. Then, by Proposition 1, there exists maximal functional uniformity U_F contained in U. By Theorem 1, the completions $(\widetilde{X},\ \widetilde{U})$ of the uniform space (X, U) are uniformly real complete, and its topological space ($\widetilde{X}$, $\widetilde{U}$)is a real complete space. We denote it by $V_U X$ and call it the Hewitt extension of the uniform space (X, U).

If U is the maximal uniformity of the Tychonoff space X, then $V_U X$ coincides with the classical Hewitt extension VX of the Tychonoff space X.

Let X be an arbitrary real complete space. By C (X) we denote the set of all continuous functions f: X → R, which generates the maximum functional uniformity of U_F. We show that the uniformity of U_F is complete. By the external characteristic, the real-complete space X is a closed subspace of the product $\prod\{(R^f, f \in C(X)\}$ of the set of copies R^f of the real line R ([2]). We denote by U the uniformity on X induced by the product $\prod\{(E_R^f, f \in C(X)\}$ the set of natural uniformities E_R^f of the real line R^f. . The uniform space (X, U) is complete as a closed subspace of the complete of the uniform space $\prod\{(R^f, E_R^f): f \in C(X)\}$. The uniformity U is generated by the restriction family by the projection $pr_f \prod\{R^f: f \in C(X)\} \to R^f$. Since $pr_f \in C(X)$, for each $f \in C(X)$, then $U \subseteq U_F$. Hense, U_F is the complete functional uniformity on X. Then the topology of the space X is also determined by this maximal functional uniformityU_F.

Definition 3. A uniform space (X, U) is called a pre-maximal functionally uniform space if its completion $(\widetilde{X},\ \widetilde{U})$ is uniformly real complete and uniformity U is an maximal functional uniformity.

Let X be an arbitrary Tychonoff space. Now we construct real complete extensions of the Tychonoff space by means of uniform structures.

We denote by V (X) the set of all pre-maximal uniformities of the Tychonoff space X. The sets V (X) are partially ordered by inclusion. We denote by H (X) the set (identifying the equivalent extension) of all real complete extensions of the Tychonoff space X. The set H (X) is also partially ordered in a natural way [2], [3].

On every real complete extension HX of the Tychonoff space X, there exists a unique complete maximal functional uniformity ϑ. It induces on X the pre-maximal functional uniformity $\vartheta \in V$ (X). Each uniformity corresponds to a unique real complete extension (H_U, X) obtained as a completion of the uniform space (X, U). It is easy to see that this correspondence between the partially ordered sets H (X) and V (X) preserves a partial order.

So, we have obtained the following theorem

Theorem 2. Partially ordered sets V (X) and H (X) are isomorphic.

Lemma. Every uniformly real complete space is uniformly homeomorphic to a closed subspace of the product of some set of copies of a real line with natural uniformity.

Theorem 3. For each uniform space (X, U) there is exactly one (up to a uniform homeomorphism) uniformly real-complete space $(\vartheta_U X, \vartheta_U)$ with the following properties:

(1) There is a uniformly homeomorphic enclosure $i: (X, U_F) \to (\vartheta_U X, \vartheta_U)$, for which $(\vartheta_U X, \vartheta_U)$ is the completion of the uniform space (X, U_F), where U_F is the maximum functional uniformity contained in U.

(2) For any continuous function $f: (X, U) \to (R, E_R)$, there is a uniformly continuous function $\tilde{f}: (\vartheta_U X, \vartheta_U) \to (R, E_R)$ such that $\tilde{f}\ i = f$.

The spaces $(\vartheta_U X, \vartheta_U)$ also satisfy the condition:

(3) For each uniformly continuous mapping f: (X, U) → (γ, M) of the uniform space (X, U) into an arbitrary uniformly real complete space (γ, M), there is a uniform mapping $\tilde{f}: (\vartheta_U X, \vartheta_U) \to (\gamma, \mathrm{M})$ such that $\tilde{f}\ i = f$.

A uniformly real complete space $(\vartheta_U, X, \vartheta_U)$ is called the Hewitt completion of the uniform space (X, U). Generally, it differs from the completion $(\tilde{X}, \tilde{U})$ of the uniform space (X,U).

Consider Lindeloff extensions on Tichonoff and uniform space.

Definition 4. Let $(X, \mathscr{U})$ be a uniform space. The uniformity $\mathscr{U}$ is called:

1) preparacompact if every cover γ of the set X such that $\gamma \cap \mathscr{F} \neq \varnothing$ for any $\mathscr{F} \in \varphi(\mathscr{U})$ belongs to $\mathscr{U}$;

2) strongly preparacompact if $\mathscr{U}$ is a preparacompact and has a base consisting of a star-finite coverings;

3) preLindeloff if $\mathscr{U}$ is a preparacompact and has a base consisting of countable coverings.

We denote by $\mathscr{U}_D(X)$ (respectively $\mathscr{U}_P(X)$, $\mathscr{U}_S(X)$, $\mathscr{U}_L(X)$) the set of all preuniversal (respectively preparacompact, strongly preparacompact, preLindeloff) uniformities of the Tychonoff space X . The sets $\mathscr{U}_D(X)$, $\mathscr{U}_P(X)$, $\mathscr{U}_S(X)$ are partially ordered by inclusion.

Theorem 4. For any Tychonoff space X the following partially ordered sets

1) $(D(X), \leq)$ and $(\mathscr{U}_D(X), \subset)$;

2) $(P(X), \leq)$ and $(\mathscr{U}_P(X), \subset)$;

3) $(S(X), \leq)$ and $(\mathscr{U}_S(X), \subset)$;

4) $(L(X), \leq)$ and $(\mathscr{U}_L(X), \subset)$.

are isomorphic.

Proposition 3. A Tychonoff space X is locally compact and paracompact (respectively Lindeloff) if and only if it contains a universal uniformity $\mathscr{U}^*$ contains a cover (countable cover respectively) consisting of compact subsets.

Remark 1. Let $(X, \mathscr{U})$-be a uniform space, and $(X, \mathscr{U})$- be its completion. If $(A, \mathscr{U}_A)$ is a precompact subspace of the space $(X, \mathscr{U})$, then the subspace $\left([A]_X, \mathscr{U}_{[A]_X}\right)$ of the space $(X, \mathscr{U})$ is compact.

Theorem 5. There is an isomorphism between the partially ordered the set of all locally compact paracompact (locally compact Lindelöff) extensions of the given Tychonoff space X and partially ordered set of all preuniversal uniformities of the space X containing a uniform cover (respectively countable uniform cover) consisting of precompact subsets.

A partially ordered set $(D(X), \leq)$ has the greatest element. This element is the extension $t_{\mathscr{U}^*} \geq X$ corresponding to the universal uniformity $\mathscr{U}^*$ of the space X. The rest partially ordered sets $(P(X), \leq)$, $(P(X), \leq)$ and $(L(X), \leq)$ generally speaking, do not have greatest elements.

Theorem 6. For Tychonoff space X the following conditions are equivalent:

(1) Partially ordered set $(P(X), \leq)$ has a greatest element.

(2) Universal uniformity U^* of the space X is preparacompact.

If U_X is a universal (the maximal) uniform of a Tychonoff space X, then a maximal locally compact paracompact (locally compact Lindeloff, respectively) extension of a uniform space (X, U_X) is a maximal locally compact paracompact (maximal locally compact Lindeloff, respectively) extension of the Tychonoff space X.

From the above results, one can get the following theorem.

Theorem 7. Among all locally compact paracompact (locally compact Lindeloff, respectively) extensions of a Tychonoff space X there is a maximal extension.

Let μX be a maximal Dieudonne complete extension of the space X and pX (spX, lX) a maximal locally compact paracompact (a maximal locally compact strongly paracompact, a maximal locally compact Lindeloff respectively) extension of the space X. Then we get the following inclusions $\mu X \subseteq pX \subseteq spX \subseteq lX \subseteq \beta X$.

If νX is a maximal real compact Hewitt extension of a space X, then the following inclusions $\mu X \subseteq \nu X \subseteq lX \subseteq \beta X$. hold.

Remark 2. Locally compact paracompact space is strongly paracompact. The locally compact strongly paracompact extensions coincide with locally compact paracompact extensions

Bibliography

1. Edwin Hewitt. Rings of real-valued continuous functions I. Transactions of the American Mathematical Society 64(1948). – P. 45-99.
2. Borubaev A.A. Ravnomernaya topologiya. Bishkek. Izdatelstvo "Ilim" 2013.
3. Aparina L.V. Hewitovskie rasshireniya blizostnych I ravnomernych prostranstv // Sibirskii mat. journal. – 1974. – T.15. – №4. – P. 707-729.
4. Engeiking R. General topology. Moskow.Mir, 1986. P.752
5. Borubaev A.A. On completions of uniform spaces and extensions of topological spaces. Bolg.Math.Spis., 1989, V.15.1. P. 63-73. (in Russian)