NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEM FOR A SECOND ORDER IMPULSIVE SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS WITH MIXED MAXIMA

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Abstract: A two-point nonlinear boundary value problem for a second order system of ordinary integro-differential equations with impulsive effects and mixed maxima is investigated. By applying some transformations is obtained a system of nonlinear functional integral equations. The existence and uniqueness of the solution of the nonperiodic two-point boundary value problem are reduced to the one valued solvability of the system of nonlinear functional integral equations in Banach space $PC([0,T],\mathbb{R}^n)$. The method of successive approximations in combination with the method of compressing mapping is used in the proof of one-valued solvability of nonlinear functional integral equations.

Keywords: Second order system, impulsive integro-differential equations, two-point nonlinear boundary value conditions, mixed maxima, successive approximations, existence and uniqueness of solution.
1. Introduction.

Mathematical model of many problems of modern sciences, technology and economics are described by differential and integro-differential equations, the solutions of which are functions with first kind discontinuities at fixed or non-fixed times. Such differential and integro-differential equations are called equations with impulsive effects. A lot of publications of studying of differential and integro-differential equations with impulsive effects, describing many natural and technical processes, are appearing (see, for example, [1–20]). Two-point and multi-point boundary value problems for the differential and integro-differential equations are studied by many authors (see, for example [21–24]). Second-order differential equations with nonlocal boundary value conditions and impulsive effects are almost not studied. The fact is that the reduction of such a problem to an equivalent functional integral equation has difficulties. In this paper, we investigate a two-point nonlinear boundary value problem for a system of second order integro-differential equations with impulsive effects, nonlinear kernel depending on construction of mixed maxima. The questions of existence and uniqueness of the solution to the nonlinear two-point boundary value problem are investigated. We note that the differential and integro-differential equations with mixed maxima have singularity in studying of the questions of solvability. Moreover, the jumpiness of solutions is a natural thing for differential equations with mixed maxima [25].

On the interval \([0, T]\) for \(t \neq t_i, \ i = 1, 2, ..., p\) we consider the questions of existence and constructive methods of calculating of the unique solutions of the second order system of nonlinear ordinary integro-differential equations with impulsive effects and maxima

\[ x''(t) = f\left(t, x(t), \int_0^T \Theta\left(t, s, \max\left\{ x(\tau) \mid \tau \in \left[\lambda_1(s) :|: \lambda_2(s)\right]\right\}\right) ds\]  

where \(t \neq t_i, \ i = 1, 2, ..., p, \ 0 = t_0 < t_1 < ... < t_p < t_{p+1} = T, \ x \in X, \ X\) is the closed bounded domain in the space \(\mathbb{R}^n, \ f(t, x, y) \in \mathbb{R}^n, \ 0 < \lambda_i(t) < T, \ i = 1, 2,\)

\[\left[\lambda_1(t) :|: \lambda_2(t)\right] = \left[\min\left\{ \lambda_1(t), \lambda_2(t)\right\};\max\left\{ \lambda_1(t), \lambda_2(t)\right\}\right], \ \max_{0 \leq s \leq T} \left[\int_0^T \Theta(t, s, x) ds\right] < \infty.\]

The equation (1) we study with nonlinear conditions

\[A_1(t)x(0^+) + B_1(t)x(T^-) = C_1(t, x(t)), \]

\[A_2(t)x'(0^+) + B_2(t)x'(T^-) = C_2(t, x(t))\]

and nonlinear impulsive effect

\[x\left(t_i^+\right) - x\left(t_i^-\right) = F_i\left(x\left(t_i\right)\right), \ i = 1, 2, ..., p,\]

\[x'\left(t_i^+\right) - x'\left(t_i^-\right) = G_i\left(x\left(t_i\right)\right), \ i = 1, 2, ..., p,\]

where \(A_i(t), B_i(t)\) are \(n \times n\)-dimensional matrix-functions, \(C_i(t, x(t)) \in \mathbb{R}^n\) is nonlinear vector-function, \(i = 1, 2, \ F_i, G_i \in \mathbb{R}^n, \ 0 < \lambda_i(t) < T, \ i = 1, 2, \ x\left(t_i^+\right) = \lim_{\nu \to 0^+} x\left(t_i + \nu\right)\), \(x\left(t_i^-\right) = \lim_{\nu \to 0^-} x\left(t_i - \nu\right)\) are right-hand side and left-hand side limits of function \(x(t)\) at the point \(t = t_i\), respectively.

\(C\left([0, T], \mathbb{R}^n\right)\) is the notation of the Banach space, which consists continuous vector function
\( x(t) \), defined on the segment \([0, T]\), with the norm

\[
\| x \| = \sqrt{\sum_{j=1}^{n} \max_{0 \leq t \leq T} |x_j(t)|},
\]

\( PC\left([0, T], \mathbb{R}^n\right) \) is the notation of the following linear vector space

\[
PC\left([0, T], \mathbb{R}^n\right) = \left\{ x : [0, T] \to \mathbb{R}^n; x(t) \in C\left((t_i, t_{i+1}], \mathbb{R}^n\right), i = 1, ..., p \right\},
\]

where \( x\left(t_i^+\right) \) and \( x\left(t_i^-\right) \) (\( i = 0, 1, ..., p \)) exist and are bounded; \( x\left(t_i^-\right) = x\left(t_i^+\right) \). Note, that the linear vector space \( PC\left([0, T], \mathbb{R}^n\right) \) is Banach space with the following norm

\[
\| x \|_{PC} = \max \left\{ \| x \|_{C((t_i, t_{i+1}))}, i = 1, 2, ..., p \right\}.
\]

2. Formulation of problem.

To find the function \( x(t) \in PC\left([0, T], \mathbb{R}^n\right) \), which for all \( t \in [0, T], t \neq t_i, i = 1, 2, ..., p \) satisfies the second-order integro-differential equation (1), nonlinear two point conditions (2), (3) and for \( t = t_i, i = 1, 2, ..., p \), \( 0 < t_1 < t_2 < ... < t_p < T \) satisfies the nonlinear limit conditions (3), (4).


Let the function \( x(t) \in PC\left([0, T], \mathbb{R}^n\right) \) is a solution of the second order boundary value problem (1)-(5). Then, integrating the integro-differential equation (1) one time on the intervals: \((0, t_1], (t_1, t_2], ..., (t_p, t_{p+1}]\), we obtain:

\[
\int_{0}^{t_{i+1}} f(x) \, ds = \int_{0}^{t_{i+1}} x'(s) \, ds = x'\left(t_i^-\right) - x'(0^+), \quad t \in (0, t_1],
\]

\[
\int_{t_i}^{t_{i+1}} f(s) \, ds = \int_{t_i}^{t_{i+1}} x'(s) \, ds = x'\left(t_{i+1}^-\right) - x'(t_i^+), \quad t \in (t_i, t_{i+1}],
\]

\[
\int_{t_p}^{t_{p+1}} f(s) \, ds = \int_{t_p}^{t_{p+1}} x'(s) \, ds = x'\left(t_{p+1}^-\right) - x'(t_p^+), \quad t \in (t_p, t_{p+1}],
\]

where for convenience, we put

\[
f(t) = f\left(t, x(t), \int_{0}^{T} \Theta(t, s, \max_{\tau \in [\lambda_1(s), \lambda_2(s)]} |x(\tau)|) \, ds\right).
\]

Hence, taking \( x'(0^+) = x'(0), \ x'(t_i^-) = x'(t_i) \) into account, on the interval \((0, T]\) we have

\[
\int_{0}^{t} f(s) \, ds = \left[ x'(t_1) - x'(0^+) \right] + \left[ x'(t_2) - x'(t_1^+) \right] + ... + \left[ x'(t) - x'(t_p^+) \right] =
\]

\[
= -x'(0) - \left[ x'(t_1^+) - x'(t_1) \right] - \left[ x'(t_2^+) - x'(t_2) \right] - ... - \left[ x'(t_p^+) - x'(t_p) \right] + x'(t).
\]

Taking into account the condition (5), the last equality we rewrite as
\[
x'(t) = x'(0) + \int_{0}^{t} f(s) \, ds + \sum_{0 < t_i < t} G_i(x(t_i)). \tag{6}
\]

We subordinate the function \( x'(t) \in PC\left([0,T], \mathbb{R}^n \right) \) in presentation (6) to satisfy the nonlinear two-point boundary condition (3):

\[
x'(T) = x'(0) + \int_{0}^{T} f(s) \, ds + \sum_{0 < t_i < T} G_i(x(t_i)). \tag{7}
\]

Substituting (7) into condition (3), we find \( x'(0) \) as follows:

\[
x'(0) = Q_2^{-1}(t) \left[ C_2(t, x(t)) - B_2(t) \int_{0}^{T} f(s) \, ds - B_2(t) \sum_{0 < t_i < T} G_i(x(t_i)) \right] \tag{8}
\]

where \( \det Q_2(t) \neq 0, \quad Q_2(t) = A_2(t) + B_2(t) \).

Substituting (8) into presentation (6), we obtain:

\[
x'(t) = Q_2^{-1}(t) \left[ C_2(t, x(t)) - B_2(t) \int_{0}^{T} f(s) \, ds - B_2(t) \sum_{0 < t_i < T} G_i(x(t_i)) \right] + \int_{0}^{t} f(s) \, ds + \sum_{0 < t_i < t} G_i(x(t_i)). \tag{9}
\]

Then, integrating integro-differential equation (9) one time on the intervals \((0,t_1], (t_1,t_2], \ldots, (t_p,t_{p+1}]\) and taking \( x'(0^+) = x'(0), \quad x'(t_{k+1}) = x'(t) \) into account, on the interval \((0,T]\) we have

\[
\int_{0}^{t} Q_2^{-1}(s) \left[ C_2(s, x(s)) - B_2(s) \int_{0}^{T} f(\theta) \, d\theta - B_2(s) \sum_{0 < t_i < T} G_i(x(t_i)) \right] \, ds + \\
+ \int_{0}^{t} \int_{0}^{s} f(\psi) \, d\psi + \sum_{0 < t_i < s} G_i(x(t_i)) \right] \, ds = \\
= \left[ x(t_1) - x(0^+) \right] + \left[ x(t_2) - x(t_1^+) \right] + \ldots + \left[ x(t_{p+1}) - x(t_p) \right] = \\
= -x(0) - \left[ x(t_1^+) - x(t_1) \right] - \left[ x(t_2^+) - x(t_2) \right] - \ldots - \left[ x(t_{p+1}) - x(t_p) \right] + x(t). \tag{10}
\]

Taking into account the nonlinear impulsive condition (4), from the last equality (10) we derive

\[
x(t) = x(0) + \int_{0}^{t} Q_2^{-1}(s) \left[ C_2(s, x(s)) - B_2(s) \int_{0}^{T} f(\theta) \, d\theta - B_2(s) \sum_{0 < t_i < T} G_i(x(t_i)) \right] \, ds + \\
+ \int_{0}^{t} \int_{0}^{s} f(\psi) \, d\psi + \sum_{0 < t_i < s} G_i(x(t_i)) \right] \, ds + \sum_{0 < t_i < t} F_i(x(t_i)). \tag{11}
\]

Applying the two-point nonlinear condition (2) to the equation (11), we find the value of \( x(0) \) as follows:
\[ x(0) = \mathcal{Q}^{-1}_1(t)C_1(t, x(t)) - \int_0^T \mathcal{Q}^{-1}_1(t)B_1(t)\mathcal{Q}^{-1}_2(s)C_2(s, x(s))ds + \]
\[ + \int_0^T \mathcal{Q}^{-1}_1(t)B_1(t)\mathcal{Q}^{-1}_2(s)B_2(s)\int f(\theta) d\theta ds + \]
\[ + \int_0^T \mathcal{Q}^{-1}_1(t)B_1(t)\mathcal{Q}^{-1}_2(s)B_2(s) \sum_{0 < t_i < t} G_i(x(t_i)) ds - \mathcal{Q}^{-1}_1(t)B_1(t) \int_0^T \int f(\theta) d\theta ds - \]
\[ - \mathcal{Q}^{-1}_1(t)B_1(t) \int_0^T \sum_{0 < t_i < t} G_i(x(t_i)) ds - \mathcal{Q}^{-1}_1(t)B_1(t) \int_0^T \sum_{0 < t_i < t} F_i(x(t_i)). \] (12)

In getting (12), we used well known formulas, which connected by the name of Dirichlet:
\[ \int_0^T g(t, s) \int f(\theta) d\theta ds = \int_0^T f(s) \int g(t, \theta) d\theta ds, \]
\[ \int_0^T g(t, s) \sum_{0 < t_i < t} I_i(x(t_i)) ds = \sum_{0 < t_i < t} \int_0^T g(t, s) ds I_i(x(t_i)). \]

Then, we rewrite (12) as follows
\[ x(0) = \mathcal{Q}^{-1}_1(t)C_1(t, x(t)) - \int_0^T \mathcal{V}_0(t, s)C_2(s, x(s))ds + \]
\[ + \int_0^T \mathcal{V}_1(t, s) f(s) ds + \sum_{0 < t_i < T} \mathcal{V}_1(t, t_i)G_i(x(t_i)) - \mathcal{Q}^{-1}_1(t)B_1(t) \sum_{0 < t_i < T} F_i(x(t_i)), \] (13)

where \( \mathcal{V}_0(t, s) = \mathcal{Q}^{-1}_1(t)B_1(t)\mathcal{Q}^{-1}_2(s), \quad \text{det} \mathcal{Q}_1(t) \neq 0, \quad \mathcal{Q}_1(t) = A_1(t) + B_1(t), \)
\[ V_1(t, s) = \mathcal{Q}^{-1}_1(t)B_1(t) \left[ \int_0^T \mathcal{Q}^{-1}_2(\theta)\left[ A_2(\theta) + 2B_2(\theta) \right] d\theta \right]. \]

Substituting (13) into presentation (11), we obtain final view of nonlinear system of functional integral equations:
\[ x(t) = J(t, x) \equiv \mathcal{Q}^{-1}_1(t)C_1(t, x(t)) + \int_0^T \mathcal{W}_0(t, s)C_2(s, x(s))ds + \]
\[ + \int_0^T \mathcal{W}_1(t, s) f(s, x(s), \Theta(s, \theta, \max \{ x(\tau) | \tau \in [\lambda_1(\theta), \lambda_2(\theta)] \}) d\theta ds + \]
\[ + \sum_{0 < t_i < T} \mathcal{W}_1(t, t_i)G_i(x(t_i)) + \sum_{0 < t_i < T} \mathcal{W}_2(t_i)F_i(x(t_i)), \] (14)

where
\[ \mathcal{W}_0(t, s) = \begin{cases} -\mathcal{V}_0(t, s), & t \leq s \leq T, \\ -\mathcal{V}_0(t, s) + \mathcal{Q}_2^{-1}(s), & 0 \leq s < t, \end{cases} \]
\[ \mathcal{W}_1(t, t_i)G_i(x(t_i)) \]
\[
W_1(t, s) = \begin{cases} 
V_1(t, s), & t \leq s \leq T, \\
V_1(t, s) - \int_0^t Q_2^{-1}(\theta) B_2(\theta) d\theta + \int_s^T Q_2^{-1}(\theta) [A_2(\theta) + B_2(\theta)] d\theta, & 0 \leq s < t,
\end{cases}
\]

\[
W_2(s) = \begin{cases} 
-Q_1^{-1}(s) B_1(s), & t \leq s \leq T, \\
Q_1^{-1}(s) A_1(s), & 0 \leq s < t.
\end{cases}
\]

3. One valued solvability.

**Theorem.** Suppose that the following conditions are fulfilled:

1. \[M_f = \max_{0 \leq t \leq T} \left| f(t, 0) \right| < \infty; \quad M_{C_j} = \max_{0 \leq t \leq T} \left| C_j(t, 0) \right| < \infty, \quad j = 1, 2;\]

2. \[m_F = \max_{i \in \{1, 2, \ldots, p\}} \left| F_i(0) \right| < \infty, \quad m_G = \max_{i \in \{1, 2, \ldots, p\}} \left| G_i(0) \right| < \infty;\]

3. For all \( t \in [0, T], \ x, y \in \mathbb{R}^n \) holds
\[
\left| f(t, x, y) - f(t, x_1, y_1) \right| \leq M_1(t) \left| x - x_1 \right| + M_2(t) \left| y - y_1 \right|;
\]

4. For all \( t, s \in [0, T]^2, \ x \in \mathbb{R}^n \) holds
\[
\left| \Theta(t, s, x) - \Theta(t, s, x_1) \right| \leq M_3(t, s) \left| x - x_1 \right|;
\]

5. For all \( t \in [0, T], \ x \in \mathbb{R}^n \) holds
\[
\left| C_j(t, x) - C_j(t, x_1) \right| \leq M_{C_j}(t) \left| x - x_1 \right|, \quad j = 1, 2;
\]

6. For all \( x \in \mathbb{R}^n, \ i = 0, 1, \ldots, p \) hold
\[
\left| F_i(x) - F_i(x_1) \right| \leq m_{F_i} \left| x - x_1 \right|, \quad \left| G_i(x) - G_i(x_1) \right| \leq m_{G_i} \left| x - x_1 \right|;
\]

7. \( \rho = \chi_1 + \ldots + \chi_5 < 1 \), where \( \chi_1, \ldots, \chi_5 \) are defined by the formulas (18)-(20) below.

Then the two-point boundary value problem (1)-(5) has a unique solution \( x(t) \in PC([0, T], \mathbb{R}^n) \). This solution can be founded by the following iterative process:

\[
\begin{align*}
\dot{x}^k(t) &= J(t, x^{k-1}), \quad k = 1, 2, 3, \\
\dot{x}^0(t) &= 0, \quad t \in (t_i, t_{i+1}), \quad i = 0, 1, 2, \ldots, p.
\end{align*}
\] \quad (15)

**Proof.** We consider the following operator \( J : PC([0, T]; \mathbb{R}^n) \rightarrow PC([0, T] \times \mathbb{R}^n) \).

Defined by the right-hand side of equation (14). Applying the principle of contracting operators to (14), we show that the operator \( J \), defined by equation (14), has a unique fixed point.

Taking first and second conditions of the theorem, for the first difference of the approximations (15) we have the following estimate
\[
\left\| x^1(t) - x^0(t) \right\| \leq \max_{0 \leq t \leq T} \left| Q_1^{-1}(t) \right| \cdot \left| C_1(t, 0) \right| + \max_{0 \leq t \leq T} \int_0^T \left| W_0(t, s) \right| \cdot \left| C_2(t, 0) \right| ds + \]
\[
+ \max_{0 \leq t \leq T} \int_0^T \left| W_1(t, s) \right| \cdot \left| f(s, \bar{\omega}, 0) \right| ds.
\]
\[ + \max_{0 \leq i \leq T} \sum_{i=1}^{p} |W_i(t, t_i) - G_i(0)| + \sum_{i=1}^{p} |W_2(t_i)| \cdot \left| F_i(0) \right| \leq \left\| Q_1^{-1}(t) \right\| M_{C_1} + \sigma_0 M_{C_2} + \sigma_1 M_f + \sigma_2 m_F < \infty, \quad (16) \]

where

\[ \sigma_0 = \max_{0 \leq i \leq T} \int_{0}^{T} |W_0(t, s)| ds, \quad \sigma_1 = \max_{0 \leq i \leq T} \int_{0}^{T} |W_1(t, s)| ds, \]
\[ \sigma_1 = \max_{0 \leq i \leq T} \sum_{i=1}^{p} |W_i(t, t_i)|, \quad \sigma_2 = \sum_{i=1}^{p} |W_2(t_i)|. \]

Then, by the third - sixth conditions of the theorem, for difference of arbitrary consecutive approximations and arbitrary \( t \in (t_i, t_{i+1}] \) we have

\[ \left\| x^{k+1}(t) - x^k(t) \right\| \leq \max_{0 \leq i \leq T} \left\| Q_1^{-1}(t) \right\| M_{A_1}(t) \left| x^k(t) - x^{k-1}(t) \right| + \]
\[ + \max_{0 \leq i \leq T} \int_{0}^{T} |W_0(t, s)| M_{A_2}(s) \left| x^k(s) - x^{k-1}(s) \right| ds + \]
\[ + \max_{0 \leq i \leq T} \int_{0}^{T} |W_1(t, s)| \left[ M_1(s) \left| x^k(s) - x^{k-1}(s) \right| \right] + \]
\[ + M_2(s) \int_{0}^{T} \left[ M_2(s) \left| x^k(s) - x^{k-1}(s) \right| \right] ds + \]
\[ + \max_{0 \leq i \leq T} \sum_{i=1}^{p} |W_i(t, t_i)| m_{2i} \left| x^k(t_i) - x^{k-1}(t_i) \right| + \sum_{i=1}^{p} |W_2(t_i)| m_{4i} \left| x^k(t_i) - x^{k-1}(t_i) \right|. \]

Hence, by the introduced norm in the space \( PC\left([0, T], R^n\right) \) we obtain

\[ \left\| x^k(t) - x^{k-1}(t) \right\|_{PC} \leq \rho \left\| x^{k-1}(t) - x^{k-2}(t) \right\|_{PC}, \quad (17) \]

where \( \rho = \chi_1 + \ldots + \chi_5, \)

\[ \chi_1 = \max_{0 \leq i \leq T} \left| Q_1^{-1}(t) \right| M_{A_1}(t), \quad \chi_2 = \max_{0 \leq i \leq T} \int_{0}^{T} |W_0(t, s)| M_{A_2}(s) ds, \quad (18) \]
\[ \chi_3 = \max_{0 \leq i \leq T} \int_{0}^{T} \left[ M_1(s) + M_2(s) \right] \int_{0}^{T} \left[ M_3(s, \theta) d\theta \right] ds, \quad (19) \]
\[ \chi_4 = \max_{0 \leq i \leq T} \sum_{i=1}^{p} |W_i(t, t_i)| m_{2i}, \quad \chi_5 = \sum_{i=1}^{p} |W_2(t_i)| m_{4i}. \quad (20) \]

According to the last condition of the theorem, we have \( \rho < 1. \) Therefore, from the estimate (17) follows that
\[ \left\| x^k(t) - x^{k-1}(t) \right\|_{PC} < \left\| x^{k-1}(t) - x^{k-2}(t) \right\|_{PC}. \]  

(21)

It implies from (21) that the operator \( J \) on the right-hand side of the equation (14) is contracting. According to fixed point principle in the Banach space \( PC([0,T], \mathbb{R}^n) \) and taking into account estimates (16), (17), we conclude that the operator \( J \) has a unique fixed point. Consequently, the two-point nonlinear boundary value problem (1)-(5) has a unique solution 

\[ x(t) \in PC([0,T], \mathbb{R}^n) \].

**REFERENCE**


15. Sharifov Ya. A. Conditions optimality in problems control with systems impulsive differential


