МАТЕМАТИКА

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NEW WEAKLY PERIODIC $p-$ADIC GENERALIZED GIBBS MEASURE FOR THE $p-$ADIC ISING MODEL ON THE CAYLEY TREE OF ORDER TWO

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***Abstract**. In the present paper, we study the $p-$adic Ising model on the Cayley tree of order two. The existence of H_A-weakly periodic (non-periodic) $p-$adic generalized Gibbs measures for this model is proved.*

***Keywords**: Cayley tree, $p-$adic numbers, $p-$adic Ising model, Gibbs measure, weakly periodic Gibbs measure.*

СУЩЕСТВОВАНИЕ СЛАБО ПЕРИОДИЧЕСКИХ ОБОБЩЕННЫХ $p-$АДИЧЕСКИХ МЕР ГИББСА ДЛЯ $p-$АДИЧЕСКОЙ МОДЕЛИ ИЗИНГА НА ДЕРЕВЕ КЭЛИ ВТОРОГО ПОРЯДКА

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***Аннотация**. В этой статье изучене $p-$адическая модель Изинга на дереве Кэли второго порядка. Доказано существование H_A - слабо периодических (непериодических) $p-$адических обобщенных мер Гиббса для этой модели.*

***Ключевые слова**: Дерево Кэли, $p-$адические числа, модель Изинга, мера Гиббса, слабо периодические мера Гиббса.*

Let Q be the field of rational numbers. For a fixed prime p, every rational number $x \neq 0$

can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in Z$, m is a positive integer, and n and m are relatively prime with p, r is called the order of x and written $r = ord_p x$. The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

This norm is non-Archimedean and satisfies the so called strong triangle inequality

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}$$

for all $x, y \in Q$.

The completion of Q with respect to the p-adic norm defines the p-adic field which is denoted by Q_p (see [1]).

The completion of the field of rational numbers Q is either the field of real numbers R or one of the fields of $p-$adic numbers Q_p (Ostrowski's theorem).

Any p-adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...)$$

where $\gamma(x) \in Z$ and the integers x_j satisfy: $x_0 \neq 0$, $x_j \in \{0,1,2,..., p-1\}$, $j \in N$ (see [1]). In this case $|x|_p = p^{-\gamma(x)}$.

The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Denote by V the set of vertices, and by L the set of edges of the Cayley tree Γ^k. Two vertices x and y are called *nearest neighbours* if there exist an edge $l \in L$ connecting them and denote by $l = \langle x, y \rangle$ (see [2]).

Fix $x_0 \in \Gamma^k$ and given vertex x, denote by $|x|$ the number of edges in the shortest path connecting x_0 and x.

For $x, y \in \Gamma^k$, denote by $d(x, y)$ the number of edges in the shortest path connecting x and y. For $x, y \in \Gamma^k$, we write $x \leq y$ if x belongs to the shortest path connecting x_0 with y, and we write $x < y$ if $x \leq y$ and $x \neq y$. If $x \leq y$ and $|y| = |x| + 1$, then we write $x \to y$.

We set

$$W_n = \{x \in V | d(x, x_0) = n\}, \ V_n = \{x \in V | d(x, x_0) \leq n\}, \ L_n = \{l = \langle x, y \rangle \in L | x, y \in V_n\}$$

$$S(x) = \{y \in V : x \to y\}, \quad S_1(x) = \{y \in V : d(x, y) = 1\}.$$

The set $S(x)$ is called *direct successor of* x.

We consider a $p-$adic Ising model where the spin values take in the set $\Phi = \{-1,1\}$. We define a configuration σ on V by the function $\sigma : x \in V \to \sigma(x) \in \Phi$. Similarly, one can be define σ_n and σ^n on V_n and W_n respectively. Ω is the set if all configuration on V and denote $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$).

For given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\varphi^{(n)} \in \Omega_{W_n}$ we define a configuration in Ω_{V_n} as follows

$$\left(\sigma_{n-1} \vee \varphi^n\right)(x) = \begin{cases} \sigma_{n-1}(x), & if \quad x \in V_{n-1} \\ \varphi^n(x), & if \quad x \in W_n \end{cases}.$$

A formal $p-$adic Hamiltonian $H:\Omega \to Q_p$ of the $p-$adic Ising model is defined by

$$H(\sigma) = J \sum_{\{x,y\}\in L} \sigma(x)\sigma(y), \tag{1}$$

where $0 < |J|_p < p^{-1/(p-1)}$ for any $\langle x, y\rangle \in L.$

We define a function $h: x \to h_x,\ \forall x \in V\setminus\{x_0\},\ h_x \in Q_P$ and consider $p-$adic probability generalized Gibbs measure μ_h^n on Ω_{V_n} defined by

$$\mu_h^{(n)}(\sigma_n) = \frac{1}{Z_n^{(h)}} \exp_p\{H_n(\sigma_n)\} \prod_{x\in W_n} h_{\sigma(x),x}, \quad n = 1,2,..., \tag{2}$$

where $Z_n^{(h)}$ is the normalizing constant

$$Z_n^{(h)} = \sum_{\varphi\in\Omega_{V_n}} \exp_p\{H_n(\varphi)\} \prod_{x\in W_n} h_{\sigma(x),x}. \tag{3}$$

A $p-$adic probability generalized Gibbs measure μ_h^n is said to be consistent if for all $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$, we have

$$\sum_{\varphi\in\Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \varphi) = \mu_h^{(n-1)}(\sigma_{n-1}). \tag{4}$$

In this case, by the $p-$adic analogue of Kolmogorov theorem there exists a unique measure μ_h on the set Ω such that $\mu_h(\{\sigma|_{V_n} \equiv \sigma_n\}) = \mu_h^{(n)}(\sigma_n)$ for all n and $\sigma_n \in \Omega_{V_n}$.(see [3])

Proposition 1.[4] The sequence of $p-$adic probability distributions $\{\mu_h^{(n)}\}_{n\geq 1}$, determined by formula (2) is consistent if and only if for any $x \in V\setminus\{x_0\}$**,** the following equation holds:

$$h_x^2 = \prod_{y\in S(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta}, \tag{5}$$

where $\theta = \exp_p(2J),\ \theta \neq 1.$

It is known that Γ^k can be represented as a non-commutative group G_k, which is the free product of k +1 cyclic groups of the second order [2].

Let $G_k / G_k^* = \{H_0, H_1, ..., H_r\}$ be a factor group, where G_k^* is a normal subgroup of index $r \geq 1$.

Definition 1. A set $h = \{h_x,\ x \in G_k\}$ of quantities is called $G_k^* -$ periodic if $h_{xy} = h_x$, for all $x \in G_k$ and $y \in G_k^*$.

For $x \in G_k$ we denote by $x_\downarrow$ the unique point of the set $\{y \in G_k : \langle x, y\rangle\} \setminus S(x).$

Definition 2. A set of quantities $h = \{h_x,\ x \in G_k\}$ is called $G_k^* -$ weakly periodic, if $h_x = h_{ij}$, for any $x \in H_i$ $x_\downarrow \in H_j$.

Definition 3. A *p*-adic generalized Gibbs measure μ is said to be $G_k^* -$(weakly) periodic if it corresponds to a $G_k^* -$ (weakly) periodic h. We call a $G_k -$periodic measure a translation-invariant measure.

Let

$$H_A = \left\{ x \in G_k : \sum_{i \in A} \omega_x(a_i) - even \right\},$$

where $\varnothing \neq A \subseteq N_k = \{1,2,3,..., k+1\}$, and $\omega_x(a_i)$ is the number of letters a_i in a word $x \in G_k$. Note that H_A- is a normal subgroup of the G_k (see [2]). Note that a weakly periodic Gibbs measure depends on normal subgroup. According to the selection of the normal subgroup, different weakly periodic Gibbs measures are found (see [3]). The set of weakly periodic Gibbs measures also includes the set of periodic (in particular translation-invariant) Gibbs measures.

We note that in the case $|A| = k+1$ (where $|A|$ is the number of elements of the set A), i.e., $A = N_k$, the concept of weak periodicity coincides with ordinary periodicity. Therefore, we consider $A \subset N_k$ such that $A \neq N_k$. In this work, we consider the case $|A| = 1$. According to (5) the H_A-weakly periodic set of h_x has the following form

$$h_x = \begin{cases} h_{00}, \ if \ \ x \in H_A, & x_\downarrow \in H_A, \\ h_{01}, \ if \ \ x \in H_A, & x_\downarrow \in G_k \setminus H_A, \\ h_{10}, \ if \ \ x \in G_k \setminus H_A, & x_\downarrow \in H_A, \\ h_{11}, \ if \ \ x \in G_k \setminus H_A, & x_\downarrow \in G_k \setminus H_A. \end{cases} \tag{6}$$

By (5) we have

$$\begin{cases} h_{00}^2 = \dfrac{\theta h_{10}^2 + 1}{\theta + h_{10}^2} \cdot \dfrac{\theta h_{00}^2 + 1}{\theta + h_{00}^2}, \\ h_{01}^2 = \left(\dfrac{\theta h_{00}^2 + 1}{\theta + h_{00}^2} \right)^2, \\ h_{10}^2 = \left(\dfrac{\theta h_{11}^2 + 1}{\theta + h_{11}^2} \right)^2, \\ h_{11}^2 = \dfrac{\theta h_{11}^2 + 1}{\theta + h_{11}^2} \cdot \dfrac{\theta h_{01}^2 + 1}{\theta + h_{01}^2}. \end{cases} \tag{7}$$

Consider operator $W : R^4 \to R^4$, defined as follows:

$$\begin{cases} h_{00}'^2 = \dfrac{\theta h_{10}^2 + 1}{\theta + h_{10}^2} \cdot \dfrac{\theta h_{00}^2 + 1}{\theta + h_{00}^2}, \\ h_{01}'^2 = \left(\dfrac{\theta h_{00}^2 + 1}{\theta + h_{00}^2} \right)^2, \\ h_{10}'^2 = \left(\dfrac{\theta h_{11}^2 + 1}{\theta + h_{11}^2} \right)^2, \\ h_{11}'^2 = \dfrac{\theta h_{11}^2 + 1}{\theta + h_{11}^2} \cdot \dfrac{\theta h_{01}^2 + 1}{\theta + h_{01}^2}. \end{cases}$$

Note that the system of (7) describes fixed points of the operator W, i.e. $h = W(h)$.

Lemma 1. The following sets are invariant with respect to the operator W:

$$I_1 = \{h \in R^4 : h_{00} = h_{01} = h_{10} = h_{11}\},$$
$$I_2 = \{h \in R^4 : h_{00} = \pm h_{11}, \ h_{10} = \pm h_{01}\}.$$

Remark 1. [4] *It is easy to see that if the function* $-h_x$ *is a solution to equation (5), then the function* $-h_x$ *is also a solution. These solutions define the same measure* μ_h *which we consider Ising model on the Cayley tree*

of order k.

We shall find H_A-weakly periodic (non-periodic) $p-$adic generalized Gibbs measure for the Ising model on the set I_2.

The system of equation (7) has the following solutions

$$h_{00_{1,2}} = \pm 1,$$
$$h_{00_{3,4}} = \pm \frac{\theta - 1 + \sqrt{(\theta+1)(\theta-3)}}{2},$$
$$h_{00_{5,6}} = \pm \frac{\theta - 1 - \sqrt{(\theta+1)(\theta-3)}}{2},$$
$$h_{00_{7,8}} = \pm \sqrt{-1}.$$

Lemma 2. The solutions h_{00_7} and h_{00_8} belong to Q_p, iff $p \equiv 1(\mathrm{mod}\,4)$.

Theorem 1. If $p \equiv 1(\mathrm{mod}\,4)$ then there exists at least one weakly periodic (non-periodic) $p-$adic generalized Gibbs measure for the $p-$adic Izing model on the Cayley tree of order two.

Remark 2. In [5] it was proved that for the Ising model on a Cayley tree of order k = 2 with respect to the normal divisor of index 2, there does not exist a weakly periodic (non-translation-invariant) Gibbs measure in real case. In p-adic case in Theorem 1 it was shown that for the Ising model there is at least one new weakly periodic p-adic generalized Gibbs measure.

REFERENCES

1. V. S. Vladimirov, I. V. Volovich and E. V. Zelenov, p -Adic Analysis and Mathematical Physics (*World Sci. Publ., Singapore*,1994).
2. U. A. Rozikov, Gibbs Measures on Cayley Trees *(World Sci. Publ., Singapore*, 2013).
3. Rozikov U. A., Rahmatullaev M. M. Description of weakly periodic Gibbs measures for the Ising model on a Cayley tree. *Theor. Math.Phys.*, 156(2): (2008).
4. Khakimov O. N. On a Generalized p-adic Gibbs Measure for Ising Model on Trees. *p-Adic Numbers, Ultrametric Anal. Appl.*, 6(3), 2014, pp.207-217.
5. Rahmatullaev M. M. "On new weakly periodic Gibbs measures of the Ising model on the Cayley tree of order 6". *J. Phys.: Conf. Ser.*, 697 (2016), 012020, pp.7.