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SOME CONSTRUCTIVE p-ADIC GENERALIZED GIBBS MEASURES FOR THE ISING MODEL ON A CAYLEY TREE

Rahmatullaev Muzaffar Muhammadjanovich, DSc., professor,
mrahmatullaev@rambler.ru
Namangan regional department, Institute of Mathematics named after V.I.Romanovsky,
Namangan, Uzbekistan,

Tukhtabaev Akbarkhuja Mamajonovich, PhD. student,
Namangan state university, Namangan, Uzbekistan,
akbarxoja.toxtaboyev@mail.ru

Abstract: *The paper is devoted to some non-periodic p-adic generalized Gibbs measures for Ising model on a semi-Cayley tree of order $k \geq 1$. We construct uncountable non-periodic p-adic generalized Gibbs measures for the Ising model on a semi-Cayley tree. We study the boundedness of the measures. Furthermore, we find conditions that guarantee existence of the phase transition.*
Keyworsds: *p-adic numbers, p-adic Ising model, Cayley tree, Gibbs measure, phase transition.*

НЕКОТОРЫЕ КОНСТРУКТИВНО *P*-АДИЧЕСКИЕ ОБОБЩЕННЫЕ МЕРЫ ГИББСА ДЛЯ МОДЕЛИ ИЗИНГА НА ДЕРЕВЕ КЭЛИ

Рахматуллаев Музаффар Мухаммаджанович, д.ф.-м.н., профессор,
mrahmatullaev@rambler.ru
Наманганское областное отделение Института математики им. В.И. Романовского, г. Наманган, Узбекистан,
Тухтабаев Акбархужа Мамажонович, PhD.докторант,
Наманганский государственный университет,
Наманган, Узбекистан,
akbarxoja.toxtaboyev@mail.ru

Аннотация: *Статья посвящена изучению некоторых непериодических p-адических обобщенных мер Гиббса для модели Изинга на полудереве Кэли порядка $k \geq 1$. Построено несчетное количество непериодических p-адических обобщенных мер Гиббса для модели Изинга на полудереве Кэли, а также изучена задача ограниченности этих мер. Кроме того, найдены условия, гарантирующие существование фазового перехода.*
Ключевые слова: *p-адические числа, p-адическая модель Изинга, дерево Кэли, мера Гиббса, фазовый переход.*

Introduction. Let $\mathbb{Q}$ be the field of rational numbers. For a fixed prime number p, every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{m}{n}$ where, $r, m, n \in \mathbb{Z}, n > 0$ and m, n are relatively prime with p. The p-adic norm of $x \in \mathbb{Q}$ is given by

$$|x|_p = \begin{cases} p^{-r}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

This norm is non-Archimedean, i.e., it satisfies the strong triangle inequality $|x+y|_p \leq \max\{|x|_p, |y|_p\}$ for all $x, y \in \mathbb{Q}$.

The completion of $\mathbb{Q}$ with respect to the p-adic norm defines the p-adic field $\mathbb{Q}_p$.

Any p-adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}\left(x_0 + x_1 p + x_2 p^2 + \ldots\right),$$

where $x_j, \gamma(x) \in \mathbb{Z}, x_0 \neq 0, x_j \in \{0,1,\ldots,p-1\}, j = 1,2,\ldots$.

In this case $|x|_p = p^{-\gamma(x)}$. For $a \in \mathbb{Q}_p$ and $r > 0$ we denote

$$B(a, r) = \{x \in \mathbb{Q}_p : |x-a|_p < r\}.$$

p-adic *exponential* is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B\left(0, p^{-\frac{1}{p-1}}\right)$.

We set

$$E_p = \left\{x \in \mathbb{Q}_p : |x-1|_p < p^{-\frac{1}{p-1}}\right\}.$$

This set is the range of the p-adic exponential function (see e.g. [2]).

Let (X, B) be a measurable space, where B is an algebra of subsets X. A function $\mu : B \to \mathbb{Q}_p$ is said to be a p-adic measure if for any $A_1, A_2, \ldots, A_n \in B$ such that $A_i \cap A_j = \varnothing, i \neq j$, the following holds:

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

A p-adic measure μ is called bounded if $\sup\{|\mu(A)|_p : A \in B\} < \infty$. It is said that p-adic measure is *probabilistic* if $\mu(X) = 1$.

Let $\Gamma_+^k = (V, L)$, be a semi-infinite Cayley tree [1] of order $k \geq 1$ with the root $x^0 \in V$ (whose each vertex has exactly $k+1$ edges, except for the root x^0, which has k edges). Here V is the set of vertices and L is the set of edges. The vertices x and y are called nearest neighbors and they are denoted by $l = \langle x, y \rangle$ if there exists an edge l connecting them. A collection of the pairs $\langle x, x_1 \rangle$, $\langle x_1, x_2 \rangle$,…, $\langle x_{d-1}, y \rangle$ is called a path from the point x to the point y. The distance $d(x,y)$, $x,y\in V$, on the semi-Cayley tree, is the number of edges of the shortest path from x to y.

We set

$$W_n=\{x\in V\,|\,d(x,x^0)=n\},\ \ V_n=\{x\in V\,|\,d(x,x^0)\le n\},$$

$$L_n=\{l=<x,y>\in L\,|\,x,y\in V_n\}.$$

The set of direct successors of $x\in W_n$ is defined by

$$S(x)=\{y\in W_{n+1}:d(x,y)=1\}.$$

We recall a coordinate structure in Γ_+^k: every vertex x (except for x^0) of Γ_+^k has coordinates $(i_1,i_2,...,i_n)$, here $i_m\in\{1,2,...,k\}$, $m=\overline{1,n}$, and for the vertex x^0 we put (0). Namely, the symbol (0) constitutes level 0, and the sites $(i_1,i_2,...,i_n)$ form level n (i.e. $d(x^0,\,x)=n$) of the lattice. Let us define on Γ_+^k binary operation $\circ:\Gamma_+^k\times\Gamma_+^k\longrightarrow\Gamma_+^k$ as follows: for any two elements $x=(i_1,i_2,...,i_n)$ and $y=(j_1,j_2,...,j_m)$ put $x\circ y=(i_1,i_2,...,i_n)\circ(j_1,j_2,...,j_m)=(i_1,i_2,...,i_n,j_1,j_2,...,j_m)$

By means of the defined operation Γ_+^k becomes a noncommutative semigroup with a unit. Let us denote this group by $(G^k,\circ)$. Using this semigroup structure one defines translations $\tau_g:G^k\to G^k$, $g\in G^k$ by $\tau_g(x)=g\circ x$.

Let $G\subset G^k$ be a sub-semigroup of G^k and $h:\ G^k\to Y$ be a Y-valued function defined on G^k. We say that h is G*-periodic* if $h(\tau_g(x))=h(x)$ for all $g\in G$ and $x\in G^k$. Any G^k-periodic function is called *translation-invariant*. Now for each $m\ge 2$ we put $G_m=\{x\in G^k:d(x,\,x^0)\equiv 0\ (\mathrm{mod}\ m)\}$. One can check that G_m is a sub-semigroup of $x\in G^k$.

We consider p-adic Ising model on the semi-infinite Cayley tree Γ_+^k. Let $\mathbb{Q}_p$ be a field of p-adic numbers and $\Phi=\{-1,1\}$. A configuration σ on $A\subseteq V$ is defined as a function $x\in A\to\sigma(x)\in\Phi$. The set of all configurations on A is denoted by $\Omega_A=\Phi^A$. For given configurations $\sigma_{n-1}\in\Omega_{V_n}$ and $\varphi^{(n)}\in\Omega_{W_n}$ we define a configuration in Ω_{V_n} as follows

$$(\sigma_{n-1}\vee\omega_n)(x)=\begin{cases}\sigma_{n-1}(x), & if\ \ x\in V_{n-1},\\ \varphi^{(n)}(x), & if\ \ x\in W_n.\end{cases}$$

A formal p-adic Hamiltonian $H:\Omega\to\mathbb{Q}_p$ of the Ising model is defined by

$$H(\sigma)=J\sum_{<x,y>\in L}\sigma(x)\sigma(y)\ , \qquad (1)$$

where $J\in B\left(0,\ p^{\frac{1}{1-p}}\right)$ for any $<x,y>\in L$.

We are aiming to study some non-periodic p-adic generalized Gibbs measures for the Ising model on a Cayley tree. Our approach is based on properties Markov random fields on the Cayley tree.

Let $h: x \in V \setminus \{x^0\} \to h_x \in \mathbb{Q}_p$ be a function. We define p-adic probability generalized Gibbs distribution $\mu_h^{(n)}$ on Ω_{V_n} by

$$\mu_h^{(n)}(\sigma_n) = \frac{1}{Z_n^{(h)}} \exp_p\{H_n(\sigma_n)\} \prod_{x \in W_n} h_x^{\sigma(x)}, \ n = 1, 2, ..., \tag{2}$$

where $Z_n^{(h)}$ is corresponding normalizing constant:

$$Z_n^{(h)} = \sum_{\sigma \in \Omega_{V_n}} \exp\{H_n(\sigma_n)\} \prod_{x \in W_n} h_x^{\sigma(x)}. \tag{3}$$

The compatibility conditions for $\mu_h^{(n)}(\sigma_n)$, $n \geq 1$ are given by the equality

$$\sum_{\varphi^{(n)} \in \Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \varphi^{(n)}) = \mu_h^{(n-1)}(\sigma_{n-1}). \tag{4}$$

In this case, by the p-adic analogue of Kolmogorov theorem ([3]) there exists a unique measure μ_h on the set Ω such that $\mu_h\left(\left\{\sigma|_{V_n} \equiv \sigma_n\right\}\right) = \mu_h^{(n-1)}(\sigma_{n-1})$, for all n and $\sigma_{n-1} \in \Omega_{V_{n-1}}$.

A limiting p-adic distribution generated by (2) is called p-adic *generalized Gibbs measure* [4].

If there are two different measures μ_s, μ_h and they are bounded, then one says that there is a *quasi-phase transition*. If there are at least two distinct p-adic generalized Gibbs measures μ and ν such that μ is bounded and ν is unbounded, then one says that a *phase transition* occurs. Moreover, if there is a sequence of sets A_n such that $A_n \in \Omega_{V_n}$ with $|\mu(A_n)|_p \to 0$ and $|\nu(A_n)|_p \to \infty$ as $n \to \infty$, then there occurs a *strong phase transition* [4].

Theorem 1. [4] *The sequence of p-adic distributions $\left\{\mu_h^{(n)}(\sigma_n)\right\}_{n \geq 1}$ determined by formula (2) is consistent if and only if for any $x \in V \setminus \{x^0\}$, the following equation holds*

$$h_x^2 = \prod_{y \in S(x)} \frac{\theta h_y^2 + 1}{h_y^2 + \theta} \tag{5}$$

where $\theta = \exp\{2J\}$, $\theta \neq 1$.

Main results. On the semi-Cayley tree of order two, we denote by $h_i^{(t)}$ ($i = 0, 1, 2$) and $h_j^{(p)}$ ($j = 1, 2$) the translation-invariant and G_2-periodic solutions of the equation (5), respectively. It is known [4] that if $p \equiv 1 \pmod 4$ then

$$h_0^{(t)} = 1, \ h_{1,2}^{(t)} = \frac{\theta - 1 \pm \sqrt{(\theta-3)(\theta+1)}}{2}, \ h_{1,2}^{(p)} = \frac{1 - \theta \pm \sqrt{(\theta-1)^2 - 4\theta^2}}{2\theta} \tag{6}$$

Let $k \geq 3$, $k_0 = 2$. For $x \in V$, by $S_{k_0}(x)$ we denote an arbitrary set of k_0 vertices of the set $S(x)$, and remaining $k - k_0$ vertices is denoted by $S_{k-k_0}(x)$. Let $k - k_0 = a + b + c$, where a and b are even, c is even or odd. We define the set of quantities $h = \{h_x, x \in V\}$ (where $h_x \in \left\{1, h_1^{(t)}, h_2^{(t)}, h_1^{(p)}, h_2^{(p)}\right\}$) as follows:

if at vertex x we have $h_x = h_i^{(t)}$ ($i = 1, 2$) ($h_x = h_i^{(p)}$ or $h_x = 1$), then the function h_y, which gives p-adic values to each vertex $y \in S(x)$ is defined by the following rule (7) (resp. (8) or (9)).

$$h_y = \begin{cases} h_i^{(t)} & \text{on} \quad a/2+2 \quad \text{vertices of} \quad S(x), \\ h_{3-i}^{(t)} & \text{on} \quad a/2 \quad \text{vertices of} \quad S(x), \\ h_i^{(p)} & \text{on} \quad b/2 \quad \text{vertices of} \quad S(x), \\ h_{3-i}^{(p)} & \text{on} \quad b/2 \quad \text{vertices of} \quad S(x), \\ 1 & \text{on} \quad c \quad \text{vertices of} \quad S(x). \end{cases} \tag{7}$$

$$h_y = \begin{cases} h_i^{(t)} & \text{on} \quad a/2 \quad \text{vertices of} \quad S(x), \\ h_{3-i}^{(t)} & \text{on} \quad a/2 \quad \text{vertices of} \quad S(x), \\ h_i^{(p)} & \text{on} \quad b/2 \quad \text{vertices of} \quad S(x), \\ h_{3-i}^{(p)} & \text{on} \quad b/2+2 \quad \text{vertices of} \quad S(x), \\ 1 & \text{on} \quad c \quad \text{vertices of} \quad S(x). \end{cases} \tag{8}$$

$$h_y = \begin{cases} h_i^{(t)} & \text{on} \quad a/2 \quad \text{vertices of} \quad S(x), \\ h_{3-i}^{(t)} & \text{on} \quad a/2 \quad \text{vertices of} \quad S(x), \\ h_i^{(p)} & \text{on} \quad b/2 \quad \text{vertices of} \quad S(x), \\ h_{3-i}^{(p)} & \text{on} \quad b/2 \quad \text{vertices of} \quad S(x), \\ 1 & \text{on} \quad c+2 \quad \text{vertices of} \quad S(x). \end{cases} \tag{9}$$

Lemma 1. *Let* $p \equiv 1 \pmod 4$. *Then any set of quantities according to the rules (7), (8) and (9) on the Cayley tree* Γ_+^k *satisfy the functional equation (5).*

Remark 1. 1) If $a=b=c=0$ in (7) and (9) then p-adic generalized Gibbs measures corresponding to set of quantities h_x are translation-invariant, the figure for case (8), we get p-adic generalized $G_2^{(2)}$-periodic Gibbs measures (see [4]);

2) If $a=b=0,\ c \neq 0$ in (9) and (7), (8) then p-adic generalized Gibbs measures corresponding to set of quantities h_x are translation-invariant (see [4]) and ART Gibbs measures, respectively (see [6]);

3) If $b=c=0,\ a \neq 0$ in (7) then p-adic generalized Gibbs measures corresponding to set of quantities h_x are (k_0)-translation-invariant (see [7]);

4) If $a=c=0,\ b \neq 0$ in (8) then p-adic generalized Gibbs measures corresponding to set of quantities h_x are (k_0)-periodic (see [7]);

5) In other cases, we get new measures except to previous known ones.

In real case Bleher-Ganikhodjaev construction was studied in [8]. We are going to investigate this construction in p-adic case. Consider an infinite path $\pi = x^0 = x_0 < x_1 < \dots$ on the semi-Cayley tree Γ_+^k (the notation $x<y$ meaning that paths from the root to y go through x). We assign the set of p-adic numbers $h^\pi = \{h_x^\pi, x \in V\}$ satisfying the equation (5) to the path π. For $n = 1, 2, \dots, x \in W_n$, the set h^π is unambiguously defined by the conditions

$$h_x^\pi = \begin{cases} \dfrac{1}{h_*}, & if \quad x \prec x_n,\ x \in W_n, \\ h_*, & if \quad x_n \prec x,\ x \in W_n, \end{cases} \tag{10}$$

where $x \prec x_n$ (resp. $x_n \prec x$) means that x is on the left (resp. right) from the path π and h_* is translation-invariant solution of the equation (5).

Lemma 2. *For any infinite path π, there exists a unique set of numbers $h^\pi = \{h_x^\pi, x \in V\}$ satisfying (5) and (10).*

Remark 2. The measure which is defined in Lemma 2, depends on the path π. The cardinality of the measures is uncountable (see [8]).

Theorem 2. *Let $p \equiv 1 \pmod 4$. Then for the measures correspond to the set of quantities according to the rules (7), (8) and (9) the followings hold*

1) If $a^2 + b^2 \neq 0$, then the measures are unbounded;

2) If $a = b = 0$, then the measures are bounded.

Theorem 3. *Let $p \geq 3$, h_x^π be the set of quantities defined by (10). Then the measures correspond to the set of quantities h_x^π are bounded if and only if $h_* = 1$.*

Theorem 4. *Let $p \geq 3$, $\mathbb{Q}_p$ be a field of p-adic numbers in which there exist translation-invariant solutions of the functional equation (5). Then there exists a phase transition in the field $\mathbb{Q}_p$.*

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