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GROUND STATES FOR THE SOS MODEL WITH COMPETING BINARY INTERACTIONS ON A CAYLEY TREE OF ORDER THREE

Rahmatullaev Muzaffar Muhammadjonovich., DSc, professor,
mrahmatullaev@rambler.ru
Institute of Mathematics named after V.I. Romanovsky
of the Academy of Sciences of the Republic of Uzbekistan,
Tashkent, Uzbekistan
Abraev Bunyod Urinboevich, PhD student
abrayev89@mail.ru
Chirchik state pedagogical university,
Chirchik, Uzbekistan.

Abstract: *We consider a SOS (solid-on-solid) model with nearest-neighbor interaction J_1, prolonged next-nearest-neighbor interaction J_2 and one level next-nearest-neighbor interaction J_3, where the spin takes values in the set $\Phi = \{0,1,2\}$ on a Cayley tree of order three. In the paper, we study translation-invariant and periodic ground states of the model SOS.*
Keywords: *Cayley tree, configuration, competing next-nearest-neighbor interactions, translation-invariant and periodic ground state.*

ОСНОВНЫЕ СОСТОЯНИЯ ДЛЯ МОДЕЛИ SOS С КОНКУРИРУЮЩИМИ БИНАРНЫМИ ВЗАИМОДЕЙСТВИЯМИ НА ДЕРЕВЕ КЭЛИ ПОРЯДКА ТРИ

Рахматуллаев Музаффар Мухаммаджанович, д.ф.-м.н., профессор,
mrahmatullaev@rambler.ru
Институт Математики имени В.И. Романовского
Академии Наук Республики Узбекистан,
Наманган, Узбекистан
Абраев Бунёд Уринбоевич, PhD докторант,
Чирчикский государственный педагогический университет,

Чирчик, Узбекистан.
abrayev89@mail.ru

***Аннотация**: Мы рассматриваем модель SOS (solid-on-solid) с взаимодействием ближайших соседей J_1, длительным взаимодействием следующих ближайших соседей J_2 и одноуровневым взаимодействием следующих ближайших соседей J_3, где спин принимает значения в множестве $\Phi=\{0,1,2\}$ на дереве Кэли порядка три. В статье исследуются трансляционно-инвариантные и периодические основные состояния модели SOS.*

***Ключевые слова**: Дерево Кэли, конфигурация, конкурирующие взаимодействия ближайших соседей, трансляционно-инвариантное и периодическое основное состояние.*

It is known that a phase diagram of Gibbs measures for a Hamiltonian is close to the phase diagram of isolated (stable) ground states of this Hamiltonian. At low temperatures, a periodic ground state corresponds to a periodic Gibbs measure (see, e.g., [1]). It leads us to investigate the problem of description of periodic and weakly periodic ground states. In this paper, we study periodic ground states for the SOS model with nearest-neighbor and competing binary interactions on a Cayley tree of order three.

Let $\Gamma^k=(V,L)$ be the Cayley tree of order $k\geq 1$, i.e., an infinite tree such that exactly $k+1$ edges are incident to each vertex. Here V is the set of vertices and L is the set of edges of Γ^k. It is known (see [2]) that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k\geq 1$ and the group G_k of the free products of $k+1$ cyclic groups $\{e,a_i\}$, $i=1,\dots,k+1$ of the second order (i.e. $a_i^2=e$, $a_i^{-1}=a_i$) with generators $a_1,a_2,\dots,a_{k+1}$.

For an arbitrary vertex $x^0\in V$, we put

$$W_n=\{x\in V\,|\,d(x,x^0)=n\}, V_n=\{x\in V\,|\,d(x,x^0)\leq n\},$$

where $d(x,y)$ is a natural distance, being the number of nearest-neighbor pairs of the minimal path between the vertices x and y. L_n denotes the set of edges in V_n. The fixed vertex x^0 is called the 0-th level and the vertices in W_n are called the n-th level. For the sake of simplicity, we put $|x|=d(x,x^0)$, $x\in V$.

Two vertices $x,y\in V$ are called the next-nearest-neighbour neighbors if $d(x,y)=2$. The next-nearest-neighbour vertices x and y are called prolonged next-nearest-neighbours if $|x|\neq|y|$ and is denoted by $\rangle x,y\langle$. The next-nearest-neighbour vertices $x,y\in V$ that are not prolonged are called one-level next-nearest-neighbours since $|x|=|y|$ and are denoted by $\rangle x,y\langle$.

For each $x\in G_k$, let $S(x)$ denote the set of direct successors of x, i.e., if $x\in W_n$ then $S(x)=\{y\in W_{n+1}: d(x,y)=1\}$. For each $x\in G_k$, let $S_1(x)$ denote the set of all neighbors of x, i.e., $S_1(x)=\{y\in G_k:\ \langle x,y\rangle\in L\}$. The set $S_1(x)\setminus S(x)$ is a singleton. Let $x_{\downarrow}$ denote the (unique) element of this set.

Let us assume that the spin values belong to the set $\Phi=\{0,1,2,...,m\}$. A function σ: $x\in V\to\sigma(x)\in\Phi$ is called configuration on V. The set of all configurations coincides with the set $\Omega=\Phi^V$.

Consider the quotient group $G_k / G_k^* = \{H_1, H_2, ..., H_r\}$, where G_k^* is a normal subgroup of index r with $r \geq 1$.

Definition 1. *A configuration* $\sigma(x),\ x \in V$ *is said to be* G_k^**-periodic, if* $\sigma(x) = \sigma_i$ *for all* $x \in H_i$. *A* G_k*-periodic configuration is called translation invariant.*

The period of a periodic configuration is the index of the corresponding normal subgroup.

The Hamiltonian of the SOS model with competing nearest-neighbour and next-nearest-neighbour binary interactions has the form:

$$H(\sigma) = -J_1 \sum_{\langle x,y\rangle \in L} |\sigma(x) - \sigma(y)| - J_3 \sum_{\substack{\rangle x,y\langle : \\ x,y \in V}} |\sigma(x) - \sigma(y)|,$$

$$-J_2 \sum_{\substack{\rangle x,y\langle : \\ x,y \in V}} |\sigma(x) - \sigma(y)| \qquad (1)$$

where $(J_1, J_2, J_3) \in \mathbb{R}^3$.

We define the energy of the configuration σ_b on b by the following formula

$$U(\sigma_b, J_1, J_2, J_3) = -\frac{1}{2} J_1 \sum_{\substack{\langle x,y\rangle : \\ x,y \in b}} |\sigma(x) - \sigma(y)| - J_2 \sum_{\substack{\rangle x,y\langle : \\ x,y \in b}} |\sigma(x) - \sigma(y)|$$

$$-J_3 \sum_{\substack{\rangle x,y\langle : \\ x,y \in b}} (|\sigma(x) - \sigma(y)|), \qquad (2)$$

where $(J_1, J_2, J_3) \in \mathbb{R}^3$.

In [4] we studied ground state for SOS model with competing nearest-neighbour and next-nearest-neighbour binary interactions on a Cayley tree of order two.

We consider the case $k = 3$.

Let $m = 2$. By (2) for any σ_b we have $U(\sigma_b) \in \{U_1, U_2, U_3, ..., U_{24}\}$, where

$$U_1 = 0, U_2 = -\frac{1}{2} J_1 - 3J_2, U_3 = -\frac{1}{2} J_1 - J_2 - 2J_3, U_4 = -J_1 - 2J_2 - 2J_3,$$

$$U_5 = -J_1 - 6J_2, U_6 = -J_1 - 2J_2 - 4J_3, U_7 = -\frac{3}{2} J_1 - 3J_2,\ U_8 = -\frac{3}{2} J_1 - J_2 - 2J_3,$$

$$U_9 = -2J_1 - 4J_2 - 4J_3, U_{10} = -3J_1 - 6J_2,\ U_{11} = -3J_1 - 2J_2 - 4J_3,$$

$$U_{12} = -2J_1 - 4J_2 - 2J_3, U_{13} = -2J_1 - 2J_2 - 4J_3,\ U_{14} = -\frac{5}{2} J_1 - 5J_2 - 2J_3,$$

$$U_{15} = -\frac{5}{2} J_1 - 3J_2 - 4J_3, U_{16} = -\frac{3}{2} J_1 - 5J_2 - 2J_3,\ U_{17} = -\frac{3}{2} J_1 - 3J_2 - 4J_3,$$

$$U_{18} = -2J_1, U_{19} = -4J_1, U_{20} = -\frac{7}{2} J_1 - 3J_2,\ U_{21} = -\frac{7}{2} J_1 - J_2 - 2J_3,$$

$$U_{22} = -\frac{5}{2} J_1 - 3J_2, U_{23} = -\frac{5}{2} J_1 - J_2 - 2J_3,\ U_{24} = -3J_1 - 2J_2 - 2J_3.$$

Definition 2. A configuration φ is called a ground state for the Hamiltonian (1), if $U(\varphi_b) = \min\{U_1, U_2, U_3, ..., U_{24}\}$ for $\forall b \in M$.

For $i = \overline{1, 24}$ we put

$$C_i = \{\sigma_b : U(\sigma_b) = U_i\} \text{ and } A_m = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; U_m = \min_{1\le k\le 24}\{U_k\}\}.$$

Quite cumbersome, but not difficult calculations show that:

$$A_1 = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \le 0; J_2 \le -\frac{1}{6}J_1; J_3 \le -\frac{1}{4}J_1 - \frac{1}{2}J_2\},$$

$$A_2 = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \le 0; J_2 = -\frac{1}{6}J_1; J_3 \le J_2\},$$

$$A_3 = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \le 0; J_2 \le -\frac{1}{6}J_1; J_3 = -\frac{1}{4}J_1 - \frac{1}{2}J_2\},$$

$$A_4 = A_{12} = A_{24} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 = 0; J_2 = 0; J_3 = 0\},$$

$$A_5 = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \le 0; J_2 \ge -\frac{1}{6}J_1; J_3 \le -\frac{1}{4}J_1 + \frac{1}{2}J_2\},$$

$$A_6 = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \le 0; J_2 \le -\frac{1}{2}J_1; J_3 \ge -\frac{1}{4}J_1 - \frac{1}{2}J_2\},$$

$$A_7 = A_{22} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 = 0; J_2 = 0; J_3 \le 0\},$$

$$A_8 = A_{23} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 = 0; J_2 \le 0; J_3 = -\frac{1}{2}J_2\},$$

$$A_9 = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 = 0; J_2 \ge 0; J_3 \ge \frac{1}{2}J_2\},$$

$$A_{10} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \ge 0; J_2 \ge \frac{1}{6}J_1; J_3 \le \frac{1}{4}J_1 + \frac{1}{2}J_2\},$$

$$A_{11} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \ge 0; J_2 \le \frac{1}{2}J_1; J_3 \ge \frac{1}{4}J_1 - \frac{1}{2}J_2\},$$

$$A_{13} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 = 0; J_2 \le 0; J_3 \ge -\frac{1}{2}J_2\},$$

$$A_{14} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \ge 0; J_2 \ge 0; J_3 = \frac{1}{4}J_1 + \frac{1}{2}J_2\},$$

$$A_{15} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \ge 0; J_2 = \frac{1}{2}J_1; J_3 \ge J_2\},$$

$$A_{16} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \le 0; J_2 \ge 0; J_3 = -\frac{1}{4}J_1 + \frac{1}{2}J_2\},$$

$$A_{17} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \le 0; J_2 = -\frac{1}{2}J_1; J_3 \ge J_2\},$$

$$A_{18} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 = 0; J_2 \le 0; J_3 \le -\frac{1}{2}J_2\},$$

$$A_{19} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \ge 0; J_2 \le \frac{1}{6}J_1; J_3 \le \frac{1}{4}J_1 - \frac{1}{2}J_2\},$$

$$A_{20} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \ge 0; J_2 = \frac{1}{6}J_1; J_3 \le J_2\},$$

$$A_{21} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 | \; J_1 \ge 0; J_2 \le \frac{1}{6}J_1; J_3 = \frac{1}{4}J_1 - \frac{1}{2}J_2\},$$

and $\bigcup_{i=1}^{24} A_i = \mathbb{R}^3$.

Let $H_A=\{x\in G_k:\sum_{i\in A}\omega_x(a_i)-\text{even}\}$, where $A\subset\{1,2,3,\dots,k+1\}$ and $\omega_x(a_i)$ is the number of a_i in the word x. If $|A|=k+1$, then $H_A\equiv G_k^{(2)}=\{x\in G_k:|x|-\text{even}\}$, where $|x|$ is length of the word x.

Note that H_A is a normal subgroup of index two (see [2]). Let $G_k/H_A=\{H_A,G_k\setminus H_A\}$ be the quotient group. Denote $H_0=H_A, H_1=G_k\setminus H_A$.

Now, we shall study H_0-periodic ground states. We note that each H_0-periodic configuration has the following form:

$$\sigma(x)=\begin{cases}\sigma_1, \text{if } x\in H_0,\\ \sigma_2, \text{if } x\in H_1,\end{cases} \qquad (3)$$

where $\sigma_i\in\Phi,\ i=1,2.$

Theorem 1. a) Let $k=3$ and $|A|=1.$ Then for the model (1) the following statements hold:

i) If $(J_1,J_2,J_3)\in A_1$ then each translation invariant configuration is a ground state.

ii) If $(J_1,J_2,J_3)\in A_2\cap A_3$ then each H_0-periodic configuration of the form (3) with $\sigma_1=\sigma_2\pm 1\ \sigma_1,\sigma_2\in\Phi,$ is a ground state.

iii) If $(J_1,J_2,J_3)\in A_5\cap A_6$ then each H_0-periodic configuration of the form (3) with $\sigma_1=\sigma_2\pm 2\ \sigma_1,\sigma_2\in\Phi,$ is a ground state.

b) Let $k=3$ and $|A|=2.$ If $(J_1,J_2,J_3)\in A_9$ then each H_0-periodic configuration of the form (3) with $\sigma_1=\sigma_2\pm 2\ \sigma_1,\sigma_2\in\Phi,$ is a ground state.

c) Let $k=3$ and $|A|=3.$ If $(J_1,J_2,J_3)\in A_{10}\cap A_{11}$ then each H_0-periodic configuration of the form (3) with $\sigma_1=\sigma_2\pm 2\ \sigma_1,\sigma_2\in\Phi,$ is a ground state.

d) Let $k=3$ and $|A|=4.$

i) If $(J_1,J_2,J_3)\in A_{18}$ then each $G_k^{(2)}$-periodic configuration of the form (3) with $\sigma_1=\sigma_2\pm 1\ \sigma_1,\sigma_2\in\Phi,$ is a ground state.

ii) If $(J_1,J_2,J_3)\in A_{19}$ then each $G_k^{(2)}$-periodic configuration of the form (3) with $\sigma_1=\sigma_2\pm 2\ \sigma_1,\sigma_2\in\Phi,$ is a ground state.

Remark 1.

1) Note that applying the methods of [3], one can construct some periodic ground states which are different from the ground states mentioned in Theorem 1.

2) Let $k=3$ and $|A|=l, l=2,3.$ If $\sigma_1=\sigma_2\pm 1,\ \sigma_1,\sigma_2\in\Phi$ then the configuration (3) is not an H_0-periodic ground state.

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