GROUND STATES FOR THE SOS MODEL WITH COMPETING BINARY INTERACTIONS ON A CAYLEY TREE OF ORDER THREE

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Abstract: We consider a SOS (solid-on-solid) model with nearest-neighbor interaction $J_1$, prolonged next-nearest-neighbor interaction $J_2$ and one level next-nearest-neighbor interaction $J_3$, where the spin takes values in the set $\Phi = \{0, 1, 2\}$ on a Cayley tree of order three. In the paper, we study translation-invariant and periodic ground states of the model SOS.

Keywords: Cayley tree, configuration, competing next-nearest-neighbor interactions, translation-invariant and periodic ground state.
Аннотация: Мы рассматриваем модель SOS (solid-on-solid) с взаимодействием ближайших соседей $J_1$, длительным взаимодействием следующих ближайших соседей $J_2$ и одноуровневым взаимодействием следующих ближайших соседей $J_3$ (где спин принимает значения в множестве $\Phi = \{0,1,2\}$) на дереве Кэли порядка три. В статье исследуются трансляционно-инвариантные и периодические основные состояния модели SOS.

Ключевые слова: Дерево Кэли, конфигурация, конкурирующие взаимодействия ближайших соседей, трансляционно-инвариантное и периодическое основное состояние.

It is known that a phase diagram of Gibbs measures for a Hamiltonian is close to the phase diagram of isolated (stable) ground states of this Hamiltonian. At low temperatures, a periodic ground state corresponds to a periodic Gibbs measure (see, e.g., [1]). It leads us to investigate the problem of description of periodic and weakly periodic ground states. In this paper, we study periodic ground states for the SOS model with nearest-neighbor and competing binary interactions on a Cayley tree of order three.

Let $\Gamma^k = (V,L)$ be the Cayley tree of order $k \geq 1$, i.e., an infinite tree such that exactly $k+1$ edges are incident to each vertex. Here $V$ is the set of vertices and $L$ is the set of edges of $\Gamma^k$. It is known (see [2]) that there exists a one-to-one correspondence between the set $V$ of vertices of the Cayley tree of order $k \geq 1$ and the group $G_k$ of the free products of $k+1$ cyclic groups $\{e, a_i\}$, $i = 1, \ldots, k+1$ of the second order (i.e. $a_i^2 = e$, $a_i^{-1} = a_i$) with generators $a_1, a_2, \ldots, a_{k+1}$.

For an arbitrary vertex $x^0 \in V$, we put

$$W_n = \{x \in V \mid d(x, x^0) = n\}, V_n = \{x \in V \mid d(x, x^0) \leq n\},$$

where $d(x, y)$ is a natural distance, being the number of nearest-neighbor pairs of the minimal path between the vertices $x$ and $y$. $L_n$ denotes the set of edges in $V_n$. The fixed vertex $x^0$ is called the 0-th level and the vertices in $W_n$ are called the $n$-th level. For the sake of simplicity, we put $|x| = d(x, x^0)$, $x \in V$.

Two vertices $x, y \in V$ are called the next-nearest-neighbour neighbours if $d(x, y) = 2$. The next-nearest-neighbour vertices $x$ and $y$ are called prolonged next-nearest-neighbours if $|x| \neq |y|$ and is denoted by $\langle x, y \rangle$. The next-nearest-neighbour vertices $x, y \in V$ that are not prolonged are called one-level next-nearest-neighbours since $|x| = |y|$ and are denoted by $\langle x, y \rangle$.

For each $x \in G_k$, let $S(x)$ denote the set of direct successors of $x$, i.e., if $x \in W_n$ then $S(x) = \{y \in W_{n+1} : d(x, y) = 1\}$. For each $x \in G_k$, let $S_1(x)$ denote the set of all neighbors of $x$, i.e., $S_1(x) = \{y \in G_k : \langle x, y \rangle \in L\}$. The set $S_1(x) \setminus S(x)$ is a singleton. Let $x_1$ denote the (unique) element of this set.

Let us assume that the spin values belong to the set $\Phi = \{0,1,2,\ldots, m\}$. A function $\sigma : x \in V \to \sigma(x) \in \Phi$ is called configuration on $V$. The set of all configurations coincides with the set $\Omega = \Phi^V$. 
Consider the quotient group $G_k / G_k^* = \{H_1, H_2, \ldots, H_r\}$, where $G_k^*$ is a normal subgroup of index $r$ with $r \geq 1$.

**Definition 1.** A configuration $\sigma(x), x \in V$ is said to be $G_k^*$-periodic, if $\sigma(x) = \sigma$, for all $x \in H_i$. A $G_k^*$-periodic configuration is called translation invariant.

The period of a periodic configuration is the index of the corresponding normal subgroup.

The Hamiltonian of the SOS model with competing nearest-neighbour and next-nearest-neighbour binary interactions has the form:

$$H(\sigma) = -J_1 \sum_{\langle x, y \rangle \in L} |\sigma(x) - \sigma(y)| - J_3 \sum_{\langle x, y \rangle \in V} |\sigma(x) - \sigma(y)|,$$

$$-J_2 \sum_{\langle x, y \rangle \in V} |\sigma(x) - \sigma(y)|$$

where $(J_1, J_2, J_3) \in \mathbb{R}^3$.

We define the energy of the configuration $\sigma_b$ on $b$ by the following formula

$$U(\sigma_b, J_1, J_2, J_3) = -\frac{1}{2} J_1 \sum_{\langle x, y \rangle \in L} |\sigma(x) - \sigma(y)| - J_2 \sum_{\langle x, y \rangle \in V} |\sigma(x) - \sigma(y)|$$

$$-J_3 \sum_{\langle x, y \rangle \in V} (|\sigma(x) - \sigma(y)|),$$

where $(J_1, J_2, J_3) \in \mathbb{R}^3$.

In [4] we studied ground state for SOS model with competing nearest-neighbour and next-nearest-neighbour binary interactions on a Cayley tree of order two.

We consider the case $k = 3$.

Let $m = 2$. By (2) for any $\sigma_b$ we have $U(\sigma_b) \in \{U_1, U_2, U_3, \ldots, U_{24}\}$, where

$$U_1 = 0, U_2 = -\frac{1}{2} J_1 - 3J_2, U_3 = -\frac{1}{2} J_1 - J_2 - 2J_3, U_4 = -J_1 - 2J_2 - 2J_3,$$

$$U_5 = -J_1 - 6J_2, U_6 = -J_1 - 2J_2 - 4J_3, U_7 = -\frac{3}{2} J_1 - 3J_2, U_8 = -\frac{3}{2} J_1 - J_2 - 2J_3,$$

$$U_9 = -2J_1 - 4J_2 - 4J_3, U_{10} = -3J_1 - 6J_2, U_{11} = -3J_1 - 2J_2 - 4J_3,$$

$$U_{12} = -2J_1 - 4J_2 - 2J_3, U_{13} = -2J_1 - J_2 - 4J_3, U_{14} = \frac{5}{2} J_1 - 5J_2 - 2J_3,$$

$$U_{15} = \frac{5}{2} J_1 - 3J_2 - 4J_3, U_{16} = -\frac{3}{2} J_1 - 5J_2 - 2J_3, U_{17} = \frac{3}{2} J_1 - 3J_2 - 4J_3,$$

$$U_{18} = -2J_1, U_{19} = -4J_1, U_{20} = -\frac{7}{2} J_1 - 3J_2, U_{21} = -\frac{7}{2} J_1 - J_2 - 2J_3,$$

$$U_{22} = \frac{5}{2} J_1 - 3J_2 - 2J_3, U_{23} = -\frac{5}{2} J_1 - J_2 - 2J_3, U_{24} = -3J_1 - 2J_2 - 2J_3.$$

**Definition 2.** A configuration $\varphi$ is called a ground state for the Hamiltonian (1), if $U(\varphi_b) = \min\{U_1, U_2, U_3, \ldots, U_{24}\}$ for $\forall b \in M$.

For $i = 1, 24$ we put
$C_i = \{ \sigma_b : U(\sigma_b) = U_i \}$ and $A_m = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid U_m = \min_{1 \leq k \leq 24} \{U_k \}\}$.

Quite cumbersome, but not difficult calculations show that:

\[ A_1 = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \leq 0; J_2 \leq -\frac{1}{6} J_1; J_3 \leq -\frac{1}{4} J_1 - \frac{1}{2} J_2 \}, \]

\[ A_2 = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \leq 0; J_2 = -\frac{1}{6} J_1 ; J_3 \leq J_2 \}, \]

\[ A_3 = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \leq 0; J_2 \leq -\frac{1}{6} J_1 ; J_3 = -\frac{1}{4} J_1 - \frac{1}{2} J_2 \}, \]

\[ A_4 = A_{24} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 = 0; J_2 = 0; J_3 = 0 \}, \]

\[ A_5 = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \leq 0; J_2 \geq -\frac{1}{6} J_1 ; J_3 \leq -\frac{1}{4} J_1 + \frac{1}{2} J_2 \}, \]

\[ A_6 = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \leq 0; J_2 \leq -\frac{1}{2} J_1; J_3 \geq -\frac{1}{4} J_1 - \frac{1}{2} J_2 \}, \]

\[ A_7 = A_{22} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 = 0; J_2 = 0; J_3 \leq 0 \}, \]

\[ A_8 = A_{23} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 = 0; J_2 \leq 0; J_3 = -\frac{1}{2} J_2 \}, \]

\[ A_9 = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \leq 0; J_2 \geq 0; J_3 \geq \frac{1}{2} J_2 \}, \]

\[ A_{10} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \geq 0; J_2 \geq \frac{1}{6} J_1; J_3 \leq \frac{1}{4} J_1 + \frac{1}{2} J_2 \}, \]

\[ A_{11} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \geq 0; J_2 \leq \frac{1}{2} J_1; J_3 \geq \frac{1}{4} J_1 - \frac{1}{2} J_2 \}, \]

\[ A_{12} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 = 0; J_2 \leq 0; J_3 \geq -\frac{1}{2} J_2 \}, \]

\[ A_{13} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 = 0; J_2 \geq 0; J_3 = \frac{1}{4} J_1 + \frac{1}{2} J_2 \}, \]

\[ A_{14} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \geq 0; J_2 \geq 0; J_3 = \frac{1}{4} J_1 + \frac{1}{2} J_2 \}, \]

\[ A_{15} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \geq 0; J_2 = \frac{1}{2} J_1; J_3 \geq J_2 \}, \]

\[ A_{16} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \leq 0; J_2 \geq 0; J_3 = -\frac{1}{4} J_1 + \frac{1}{2} J_2 \}, \]

\[ A_{17} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \leq 0; J_2 = -\frac{1}{2} J_1; J_3 \geq J_2 \}, \]

\[ A_{18} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 = 0; J_2 \leq 0; J_3 \leq -\frac{1}{2} J_2 \}, \]

\[ A_{19} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \geq 0; J_2 \leq \frac{1}{6} J_1; J_3 \leq \frac{1}{4} J_1 - \frac{1}{2} J_2 \}, \]

\[ A_{20} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \geq 0; J_2 = \frac{1}{6} J_1; J_3 \leq J_2 \}, \]

\[ A_{21} = \{(J_1, J_2, J_3) \in \mathbb{R}^3 \mid J_1 \geq 0; J_2 \leq \frac{1}{6} J_1; J_3 = \frac{1}{4} J_1 - \frac{1}{2} J_2 \}, \]

and $\bigcup_{i=1}^{24} A_i = \mathbb{R}^3$. 

Let $H_A = \{ x \in G_k : \sum_{a \in A} \omega_a(a) = \text{even} \}$, where $A \subset \{1, 2, 3, \ldots, k+1\}$ and $\omega_a(a)$ is the number of $a_i$ in the word $x$. If $|A| = k+1$, then $H_A \equiv G_k^{(2)} = \{ x \in G_k : |x| = \text{even} \}$, where $|x|$ is length of the word $x$.

Note that $H_A$ is a normal subgroup of index two (see [2]). Let $G_k / H_A = \{ H \} \equiv G_k \setminus H_A$ be the quotient group. Denote $H_0 = H_A, H_1 = G_k \setminus H_A$.

Now, we shall study $H_0$-periodic ground states. We note that each $H_0$-periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_1, & \text{if } x \in H_0, \\ \sigma_2, & \text{if } x \in H_1, \end{cases}$$

(3)

where $\sigma_i \in \Phi, i = 1, 2$.

**Theorem 1.** a) Let $k = 3$ and $|A| = 1$. Then for the model (1) the following statements hold:

i) If $(J_1, J_2, J_3) \in A_1$ then each translation invariant configuration is a ground state.

ii) If $(J_1, J_2, J_3) \in A_2 \cap A_3$ then each $H_0$-periodic configuration of the form (3) with $\sigma_1 = \sigma_2 \pm 1, \sigma_1, \sigma_2 \in \Phi$, is a ground state.

iii) If $(J_1, J_2, J_3) \in A_5 \cap A_6$ then each $H_0$-periodic configuration of the form (3) with $\sigma_1 = \sigma_2 \pm 2, \sigma_1, \sigma_2 \in \Phi$, is a ground state.

b) Let $k = 3$ and $|A| = 2$. If $(J_1, J_2, J_3) \in A_9$ then each $H_0$-periodic configuration of the form (3) with $\sigma_1 = \sigma_2 \pm 2, \sigma_1, \sigma_2 \in \Phi$, is a ground state.

c) Let $k = 3$ and $|A| = 3$. If $(J_1, J_2, J_3) \in A_0 \cap A_1$ then each $H_0$-periodic configuration of the form (3) with $\sigma_1 = \sigma_2 \pm 2, \sigma_1, \sigma_2 \in \Phi$, is a ground state.

d) Let $k = 3$ and $|A| = 4$.

i) If $(J_1, J_2, J_3) \in A_8$ then each $G_k^{(2)}$-periodic configuration of the form (3) with $\sigma_1 = \sigma_2 \pm 1, \sigma_1, \sigma_2 \in \Phi$, is a ground state.

ii) If $(J_1, J_2, J_3) \in A_9$ then each $G_k^{(2)}$-periodic configuration of the form (3) with $\sigma_1 = \sigma_2 \pm 2, \sigma_1, \sigma_2 \in \Phi$, is a ground state.

**Remark 1.**

1) Note that applying the methods of [3], one can construct some periodic ground states which are different from the ground states mentioned in Theorem 1.

2) Let $k = 3$ and $|A| = l, l = 2, 3$. If $\sigma_1 = \sigma_2 \pm 1, \sigma_1, \sigma_2 \in \Phi$ then the configuration (3) is not an $H_0$-periodic ground state.

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