МАТЕМАТИКА

УДК 517.97 ББК 22.18

https://doi.org/10.52754/16948645_2023_2_169

OPTIMAL NUMBER OF PURSUERS IN THE GAME ON THE 1-SKELETON OF TESSERACT

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Abstract: *In the paper, we study pursuit and evasion differential games within a four-dimensional cube, where all the players move along the edges. The problem is to find the optimal number of pursuers in the game, to construct strategies for the pursuers in pursuit game, and evasion strategy in evasion game.*
Keywords: *tesseract, differential game, pursuer, evader, strategy.*

ОПТИМАЛЬНОЕ КОЛИЧЕСТВО ПРЕСЛЕДОВАТЕЛЕЙ В ИГРЕ НА РЕБЕРНОМ ОСТОВЕ ТЕССЕРАКТА

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Аннотация: *В статье изучаются дифференциальные игры преследования и уклонения внутри четырехмерного куба т.е. тессеракта, где все игроки перемещаются по ребрам. Задача состоит в том, чтобы найти оптимальное количество преследователей в игре, построить стратегии преследователей в игре преследования и стратегию уклонения в игре уклонения.*
Ключевые слова: *тессеракт, дифференциальная игра, преследователь, убегающий, стратегия.*

1. Introduction

The notion of differential game was introduced by Isaacs [1]. Pontryagin [2] and Krasovskii [3] gave fundamental contribution to the theory of differential games creating the formalizations to the theory. The theory was further developed by many researchers such as Azamov [4], Berkovitz [5], Elliott and Kalton [6], Fleming [7], Friedman [8], Hajek [9], Mishchenko [10], Petrosyan [11], Satimov [12] and others.

One of the most significant current discussions in differential games is multi player differential games. In the last three decades of the past century there had seen the rapid development of differential games of many players such as [13, 14].

In recent years, there has been an increasing interest in differential games of several players (see for example [15, 16, 17, 18, 19, 20, 21, 22])

However, if exhaustable resources such as energy, fuel, resources etc. are restricted for the modeling control processes, then control functions are restricted by integral constraints. The method of resolving functions for the games with integral constraints was developed by Belousov [23], to obtain a sufficient condition in solving a pursuit differential game. The solution was then extended to the case of convex integral constraints [24]. Other works in differential game in $\mathbb{R}^n$ of integral constraints include [19, 20, 21, 25].

Some games with either geometric or integral constraint, restrict the movement of players to some specific state constraints. For examples, differential games in a convex subset of $\mathbb{R}^n$ were studied by [26, 27, 28, 29] and [30]. Furthermore, differential games within a geometrical structure in the form of abstract graphs as its state constraint, are of increasing interest.

These types of games have minimax forms of which, each being a model for the search problem of a moving object, as mentioned in [31, 32]. It could be called multi-move games as in the work of [33, 34]. In this type of games, players move from one vertex to its adjacent vertex by jumping constitute one type. Another type of game on abstract graphs is where players move along the edges of a given graph embedded in a Euclidean space, as studied in [16, 32, 35, 36, 37, 38].

One of the most recent work involved a study in a differential game of many pursuers and one evader within 1-skeleton graph of an orthoplex of dimension $d+1$, as discussed in [39]. Both pursuit and evasion games were considered on the edge graph K_{d+1} of the orthoplex $\Sigma^{2(d+1)}$ in the Euclidean space $\mathbb{R}^{d+1}$. It was shown that pursuit can be completed in the case of $n=k=d+1$, or when $n \neq k$ and $n+k \geq 2d$. Otherwise, it was proven that evasion is possible.

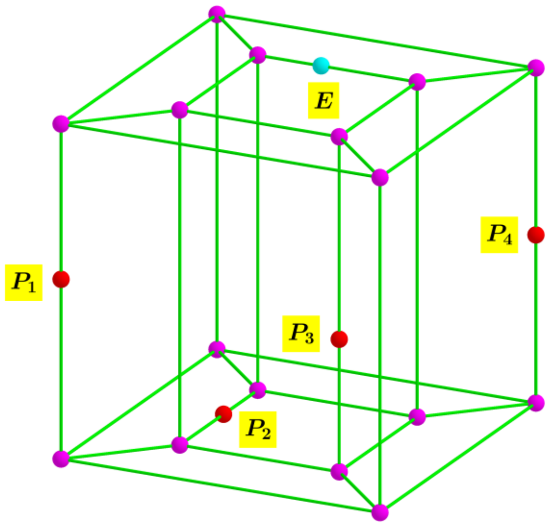

Figure 1: The graph of four dimensional cube K.

The current paper intends to study both pursuit and evasion differential games within a four-dimensional cube. All the players move along the edges of the cube and the search for the optimal number of pursuers to ensure pursuit can be completed, are also done.

2. Statement of problem

We consider a differential game of n pursuers x_1, x_2,..., x_n, $n \geq 2$, and one evader y whose dynamics are given by the following equations

$$
\begin{aligned}
&\dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \ i = 1,...,n,\\
&\dot{y} = v, \quad\ \ y(0) = y_0,
\end{aligned}
\tag{1}
$$

where $x_{i0}, y_0 \in K$, $x_{i0} \neq y_0$, $i = 1,...,n$; u_i is the control parameter of i-th pursuer, and v is the control parameter of the evader. All the players move along the edges of four-dimensional cube K. The maximal speeds of the pursuers x_1, x_2,..., x_n are ρ_1, ρ_2,..., ρ_n, respectively, and that of evader y is 1, i.e., $|u_i| \leq \rho_i$, $i = 1,...,n$, $|v| \leq 1$. It is assumed that $1/3 \leq \rho_i < 1$.

We let $B(r)$ denote the ball of radius r and centered at the origin of the Euclidean space $\mathbb{R}^{d+1}$.

Definition 1. *A measurable function* $u_i(\cdot)$, $u_i : [0,\infty) \to B(\rho_i)$ *is called admissible control of the* i*-th pursuer,* $i \in \{1,...,n\}$, *if for the solution* $x_i(\cdot)$ *of the equation*

$$\dot{x}_i = u_i, \, x_i(0) = x_{i0},$$

we have $x_i(t) \in K$, $t \geq 0$.

Definition 2. *A measurable function* $v(\cdot)$, $v : [0,\infty) \to B(\sigma)$ *is called admissible control of the evader, if for the solution* $y(\cdot)$ *of the equation*

$$\dot{y} = v, \, y(0) = y_0,$$

we have $y(t) \in K$, $t \geq 0$.

We consider pursuit and evasion differential games. In the pursuit differential game pursuers apply some strategies and evader uses an arbitrary admissible control. Let us define strategies of pursuers.

Definition 3. *The functions* $(t, x_1,..., x_n, y, v) \to U_i(t, x_1,..., x_n, y, v)$, $i = 1,2,...,n$, *are called strategies of pursuers* x_i, $i = 1,2,...,n$, *if the initial value problem (1) has a unique solution* $x_1(t),..., x_n(t)$, $y(t) \in K$, $t \geq 0$, *for* $u_i = U_i(t, x_1,..., x_n, y, v)$, $i = 1,2,...,n$, *and for any admissible control* $v = v(t)$ *of the evader.*

Definition 4. *If, for some number* $T > 0$, *there exist strategies of pursuers such that* $x_i(\tau) = y(\tau)$ *at some* τ, $0 < \tau \leq T$ *and* $i \in \{1,...,n\}$, *then pursuit is said to be completed. The pursuers are interested in completing the pursuit as earlier as possible.*

Definition 5. *A function* $(t, x_1,..., x_n, y) \to V(t, x_1,..., x_n, y)$ *is called a strategy of the evader* y *if the initial value problem (1) has a unique solution* $x_1(t),..., x_n(t)$, $y(t) \in K$, $t \geq 0$, *for* $v = V(t, x_1,..., x_n, y)$ *and for any admissible controls of pursuers* $u_i = u_i(t)$, $i = 1,2,...,n$.

Definition 6. *If, for some initial states of players* $x_{10},..., x_{n0}, y_0 \in K$, *there exists a strategy of evader such that* $x_i(t) \neq y(t)$ *for all* $t \geq 0$, *and* $i = 1,2,...,n$, *then we say that evasion is possible in the game in* K.

The evader is interested in maintaining the inequality $x_i(t) \neq y(t)$ as long as possible. Since for some initial states the evader may be trapped by pursuers and pursuit can be completed by pursuers easily, therefore this definition contains the phrase "for some initial states of players $x_{10},..., x_{n0}, y_0 \in K$".

The number $N = N(K)$ is called the optimal number of pursuers for the game on the cube K if, for any initial states of players, pursuit can be completed in the game with N pursuers and evasion is possible in the game with $N-1$ pursuers.

The problem is to find the optimal number of pursuers N in the game, to construct strategies for the pursuers in pursuit game, and evasion strategy in evasion game.

3. Main Result

Without any loss of generality we assume that the lengths of edges of the cube K is equal to 1.

3.1 Pursuit differential game. In this subsection, we prove the following statement.

Theorem 1. *Four pursuers x_1, x_2, x_3, x_4 can complete the pursuit in the differential game on 1-skeleton of the four dimensional cube K.*

Proof.

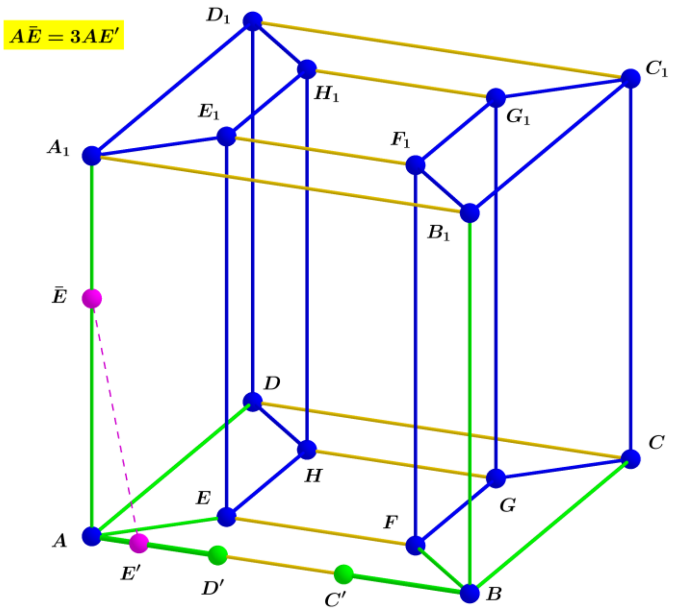

Figure 2: The shadow E' of the point $\bar{E} \in AA_1$: $A\bar{E} = 3AE'$.

Let the points D' and C' divide the edge AB into three equal segments: $AD' = D'C' = C'B$ (Figure 2). To construct strategies of pursuers, we define the shadow $E' \in AB$ of the evader $\bar{E} \notin AB$ on the edge AB as follows.

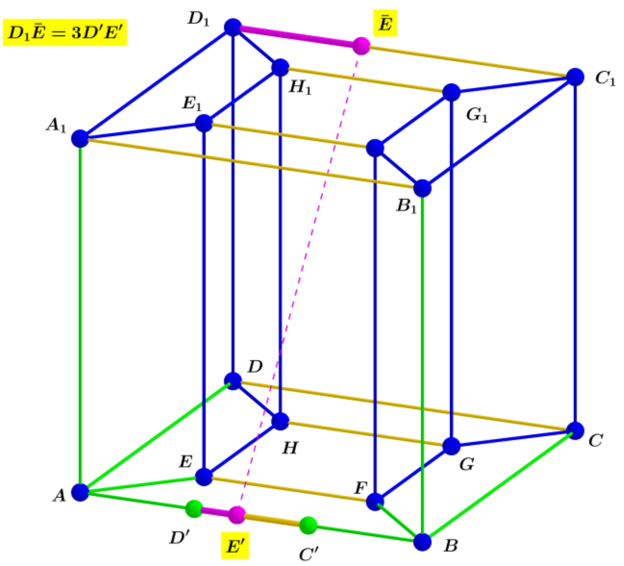

Figure 3: The shadow E' of the point $\bar{E}$: $D_1\bar{E} = 3D'E'$.

1. If $\bar{E} \in AE$ or AD or AA_1 (these edges are highlighted in green in Figure 2), then $A\bar{E} = 3AE'$.

2. If $\bar{E} \in BC$ or BF or BB_1 (these edges are highlighted in green), then $B\bar{E} = 3BE'$.

3. If $\bar{E} \in HE$ or HD or HH_1 or DD_1 or EE_1 or A_1D_1 or D_1H_1 or H_1E_1 or A_1E_1 (these edges are highlighted in blue), then $E' = D'$.

4. If $\bar{E} \in GF$ or GC or GG_1 or FF_1 or CC_1 or B_1C_1 or C_1G_1 or G_1F_1 or F_1B_1 (these edges are highlighted in blue), then $E' = C'$.

5. If $\bar{E}$ is on the edge parallel to AB, that is, $\bar{E} \in EF$ or HG or DC or A_1B_1 or E_1F_1 or H_1G_1 or D_1C_1 (these edges are highlighted in gold), then $E' \in C'D'$ is defined from the condition that the distance of $\bar{E}$ from the left end point of the edge that contains $\bar{E}$ is equal to $3D'E'$ (Figure 3).

Since the maximum speed of the evader is 1, the speed of the point E' doesn't exceed 1/3. If the pursuer P_1 moves from the vertex A to the vertex B along the edge AB, then P_1 coincides with either the real evader $\bar{E}$ or its shadow E'. If P_1 coincides with the real evader $\bar{E}$, then pursuit is completed. If P_1 coincides with the shadow of evader E', then P_1 can further move on the point E' holding this point. Then, as the evader $\bar{E}$ reaches one of the vertices A and B at some time, we have $P_1 = E' = \bar{E}$ at that time, that is, the evader is captured at that time. Thus, starting from the time when $P_1 = E'$ pursuer P_1 can guard the edge AB from the evader.

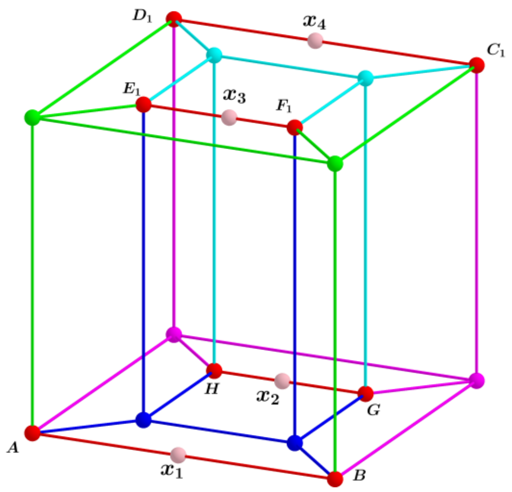

Figure 4: The edges controlled by the pursuers of the 4D cube

We construct now strategies for the pursuers. Let pursuers x_1, x_2, x_3, and x_4 come to the vertices A, H, E_1, and D_1, respectively. Next, the pursuers x_1, x_2, x_3, and x_4 move along the edges AB, HG, E_1F_1, and D_1C_1, respectively, and catch the shadows of the evader on these edges, respectively. Each pursuer starting from the time when he catches the shadow of the evader moves holding the shadow of the evader on that edge.

Let all the pursuers x_1, x_2, x_3, x_4 catch the shadows of the evader on the edges AB, HG, E_1F_1, D_1C_1, respectively, by the time T (Figure 4).

Then at the time T the evader is on one of the edges colored in Green or Blue or Cyan or Magenta (Figure 4). In each case, the evader is trapped by three pursuers and cannot walk from one edge to another edge of distinct colors.

Without any loss of generality, we assume that the evader is on a green edge. Then it is trapped by the pursuers x_1, x_3, x_4. Then, we let the pursuers x_1, x_3, x_4 control the edges AB, E_1F_1, D_1C_1, respectively, holding the evader's shadow and let the pursuer x_2 move towards the evader. Since the green edges form a tree, therefore the pursuer x_2 catches the evader or forces it to reach one of the edges AB, E_1F_1, D_1C_1. In the latter case, the evader will be caught by one of the pursuers x_1, x_3, x_4. The proof of the theorem is complete.

3.2 Evasion differential game. We prove now the following evasion possible statement.

Theorem 2. *Evasion from three pursuers x_1, x_2, x_3 is possible in the differential game on 1-skeleton of the four dimensional cube K.*

Proof. To prove this theorem, we show that, for some initial states of players, there exist a strategy of the evader such that evasion is possible.

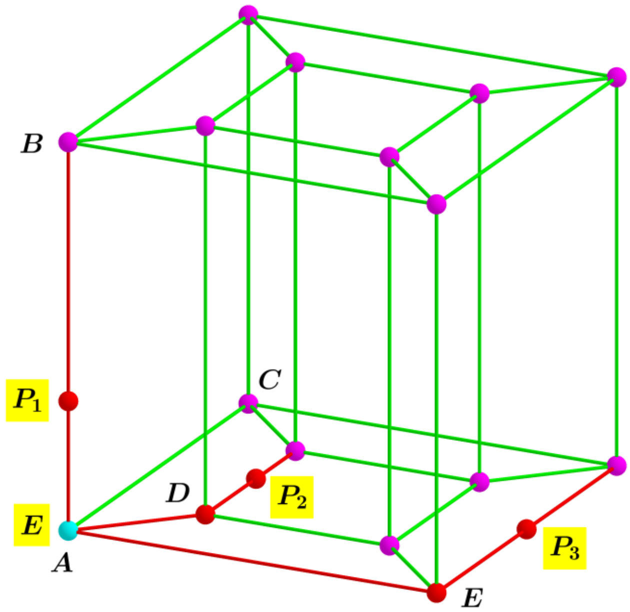

Figure 5: The shadow E' of the point $\bar{E}$: $D_1\bar{E} = 3D'E'$.

Let the evader is at some vertex of the cube say at the vertex A and any pursuer is not at the vertex A. We show that from such initial positions of players evasion is possible. We construct a strategy for the evader as follows. The evader stays at the point A until the distance between the point A and closest to this point pursuer becomes less than or equal to 1/3 at some time t_1. Note that it is possible that $t_1 = 0$. For example, if the distance of a pursuer from the point A is less than or equal to 1/3 at the initial time, then, clearly, $t_1 = 0$.

For the definiteness, we assume that the neighboring to A vertices of the cube are B, C, D, E, and $AP_1 = 1/3$ at some time $t_1 \geq 0$ and that the pursuer P_1 is on the edge AB (Figure 5). Since the distance between any two of the points C, D, E along the 1-skeleton of the cube is greater or equal to 2, and the speeds of pursuers P_2 and P_3 are less than or equal to 1, therefore these pursuers can reach only one of the vertices C, D, E for the unit time. Clearly, the pursuer P_1 cannot reach these vertices for the unit time. Hence, the evader can reach one of these vertices, say the vertex C for the unit time not being captured by the pursuers. Thus, the evader is at the vertex C at the time $t_1 + 1$. We repeatedly use this reasoning, to conclude that evasion is possible on the infinite time interval $[0, \infty)$. The proof of the theorem is complete.

4. Conclusion

We have studied pursuit and evasion differential games on the edge graph of four dimensional cube. We have established that in the differential game of four pursuers and one evader pursuit can be completed. Here, one of the central results of the paper is the construction of strategies for the pursuers. Next, we proved that in the differential game of three pursuers and one evader evasion is possible. Also, we have proposed a strategy for the evader that ensures evasion in the differential game.

Based on the two theorems proved for the pursuit and evasion differential games we can conclude that the optimal number of pursuers N in the game is $N = 4$.

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