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ON THE BLOWING-UP OF SOLUTIONS OF ONE DEGENERATE CROSS-WISE SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract: *This paper is devoted to the cross-wise system of nonlinear parabolic equations with nonlinear boundary conditions. Standard equation's method used in order to obtain critical global existence curve for the problem. Self-similar subsolutions constructed to show the blowing-up of solutions in finite time.*
Keywords: *cross-wise system, blow-up solution, nonlinear boundary condition, global existence, critical exponent.*

О НЕОГРАНИЧЕННЫХ РЕШЕНИЯХ ОДНОЙ ВЫРОЖДАЮЩЕЙСЯ КРЕСТ-НАКРЕСТ СИСТЕМЫ С НЕЛИНЕЙНЫМИ ГРАНИЧНЫМИ УСЛОВИЯМИ

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Аннотация: *Данная работа посвящена крест-накрест системе нелинейных параболических уравнений с нелинейными граничными условиями. Использовано метод стандартного уравнения для получения критического кривого глобального существования решения. Построены автомодельные субрешения для демонстрации неограниченных решений за конечное время.*
Ключевые слова: *cross-wise system, blow-up solution, nonlinear boundary condition, global existence, critical exponent.*

Consider the following cross-wise system

$$\begin{aligned}\frac{\partial u}{\partial t} &= v^{\alpha_1}\frac{\partial}{\partial x}\left(u^{m_1-1}\left|\frac{\partial u^{k_1}}{\partial x}\right|^{p_1-2}\frac{\partial u}{\partial x}\right)\\ \frac{\partial v}{\partial t} &= u^{\alpha_2}\frac{\partial}{\partial x}\left(v^{m_2-1}\left|\frac{\partial v^{k_2}}{\partial x}\right|^{p_2-2}\frac{\partial v}{\partial x}\right)\end{aligned} \tag{1}$$

with nonlinear boundary

$$\begin{aligned}-u^{m_1-1}\left|\frac{\partial u^{k_1}}{\partial x}\right|^{p_1-2}\frac{\partial u}{\partial x}\Bigg|_{x=0} &= v^{q_1}(0,t)\\ -v^{m_2-1}\left|\frac{\partial v^{k_2}}{\partial x}\right|^{p_2-2}\frac{\partial v}{\partial x}\Bigg|_{x=0} &= u^{q_2}(0,t)\end{aligned} \tag{2}$$

and initial conditions

$$\begin{aligned}u(x,0) &= u_0(x)\\ v(x,0) &= v_0(x)\end{aligned} \tag{3}$$

where parameters $0<\alpha_i<1,\ m_i>1,\ k_i>1,\ p_i>2,\ q_i>0\ (i=1,2)$ and $u_0,\ v_0$ are nonnegative continuous functions with compact support in R_+.

This system has been proposed as a mathematical model for a variety of physical problems, for example, this system can be used to describe the development of multiple groups in the dynamics of biological groups, where u and v are the densities of different groups [1],[6]-[8].

In some cases, this system is closer to real-world conditions that the classical divergent form of the system. For instance, for biological species, divergent distribution means that the species can move to all locations within its environment with equal probability. However, if we consider this problem with objective conditions, population density will affect the propagation rate. Therefore, a kind of diffusion equation will be more realistic. For this type of diffusion, propagation rate is governed by population density, which increases for large populations and decreases for small populations equation [9]-[10].

Aripov and Rakhmonov [5] considered the problem

$$\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\frac{\partial u^m}{\partial x}\right), \quad (x,t)\in R_+\times(0,+\infty)$$

$$-\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\frac{\partial u^m}{\partial x}(0,t) = u^q(0,t), \quad t\in(0,+\infty)$$

$$u(0,x) = u_0(x), \quad x\in R_+$$

where $\rho(x)=(1+x)^n$, $m>0$, $1<p<1+1/m$, $q>0$, $n\in R$. For the critical case critical exponents of the global existence and Fujita type of solutions are obtained as $q_0=\dfrac{(m(n+1)+1)(p-1)}{p+n}$, $q_c=m(p-1)+\dfrac{p-1}{n+1}$ by application of the self-similar solution of the form

$$u(x,t)=e^{Lt}g(\xi),\quad g(\xi)=\left(K+e^{-M\xi^{\frac{p+n}{p-1}}}\right)^{\frac{1}{m}},\quad \xi=(1+x)e^{Jt}.$$

The leading term of the asymptotic behaviour of self-similar solutions of the problem is obtained. On the basis of the asymptotic of solutions, suitable initial approximations are offered for the iterative process in the case of fast diffusion, depending on the values of the numeric parameters.

Positive solutions of degenerate and strongly coupled quasilinear parabolic system

$$u_t=v^{\alpha}\Delta u+u\left(a_1-b_1u^l+c_1v^s\right)$$
$$v_t=u^{\beta}\Delta v+v\left(a_2-b_2v^p+c_2v^q\right)$$

with null Dirichlet boundary condition describing a cooperating model with crosswise diffusion, where the constants $a_i,b_i,c_i>0$ ($i=1,2$), $\alpha,\beta\geq 0$ and $l,s,p,q\geq 1$ studied in [3]. Local existence of positive classical solution is proved. Moreover, it will be proved that the solution is global if intra-specific competitions of the species are strong, whereas the solution may be non-global if the inter-specific cooperation is strong and $0<\alpha\leq s$, $0<\beta\leq p$ with $\alpha,\beta\leq 2$.

In [2] following non-linear degenerate parabolic system with Dirichlet boundary condition is studied

$$u_t=v^{\alpha_1}\left(u_{xx}+au\right)$$
$$v_t=u^{\alpha_2}\left(v_{xx}+bv\right)$$

The regularization method and the upper and lower solution technique are used to show the local existence of a solution for a non-linear degenerate parabolic system. The existence of a global solution is discussed, the blow-up property of the solution is set.

The work of the authors Chen Botao, Mi Yongsheng, Mu Chunlai [4] is devoted to the study of conditions for global solvability and nonsolvability in time of solutions to the following problem

$$u_t=\left(\left|u_x\right|^{p_1}\left(u^{m_1}\right)_x\right)_x,\quad v_t=\left(\left|v_x\right|^{p_2}\left(v^{m_2}\right)_x\right)_x,\quad x>0,\ 0<t<T,$$
$$-\left|u_x\right|^{p_1}\left(u^{m_1}\right)_x(0,t)=u^{\alpha_1}(0,t)v^{\beta_2}(0,t),\quad 0<t<T,$$
$$-\left|v_x\right|^{p_2}\left(v^{m_2}\right)_x(0,t)=u^{\alpha_2}(0,t)v^{\beta_1}(0,t),\quad 0<t<T,$$
$$u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad x>0$$

where $m_i\geq 1$, $p_i>0$, $q_i>0$, $\alpha_i>0$, $\beta_i>0$. Critical exponents were obtained for problem.

Motivated by the above mentioned works, the aim of this paper is to construct the self-similar subsolutions to show the blow-up in finite time solutions of the problem (1)-(3).

We need the following notation

$$s_1=m_1+k_1(p_1-2),\ s_2=m_2+k_2(p_2-2),$$
$$\gamma_1=\frac{(p_2-1)(p_1q_1+\alpha_1(p_1-1))+(p_1-1)(s_2+p_2-1)}{(p_1q_1+\alpha_1(p_1-1))(p_2q_2+\alpha_2(p_2-1))-(s_1+p_1-1)(s_2+p_2-1)},$$

$$\gamma_2=\frac{(p_1-1)(p_2q_2+\alpha_2(p_2-1))+(p_2-1)(s_1+p_1-1)}{(p_1q_1+\alpha_1(p_1-1))(p_2q_2+\alpha_2(p_2-1))-(s_1+p_1-1)(s_2+p_2-1)},$$

$$\sigma_1=\frac{\gamma_2q_1-\gamma_1s_1}{p_1-1},\ \sigma_2=\frac{\gamma_1q_2-\gamma_2s_2}{p_2-1}.$$

Theorem. *Let*

$$\min\left\{\frac{\alpha_1(p_2-1)-(p_1-1)(s_2-1)}{\alpha_1\alpha_2-(s_1-1)(s_2-1)},\frac{\alpha_2(p_1-1)-(p_2-1)(s_1-1)}{\alpha_1\alpha_2-(s_1-1)(s_2-1)}\right\}>0.$$

If $(p_1q_1+\alpha_1(p_1-1))(p_2q_2+\alpha_2(p_2-1))>(s_1+p_1-1)(s_2+p_2-1)$ *then the system (1)-(3) has a solution that blows-up in finite time.*

Proof. To prove the nonexistence of global solutions, we construct a blow-up self-similar solution of the system. Construct

$$\begin{aligned}\underline{u}(x,t)&=(T-t)^{-\gamma_1}\ f_1(\xi_1),\quad \xi_1=x(T-t)^{-\sigma_1}\\ \underline{v}(x,t)&=(T-t)^{-\gamma_2}\ f_2(\xi_2),\quad \xi_2=x(T-t)^{-\sigma_2}\end{aligned},\tag{4}$$

where T is a positive constant and f_1,f_2 are two compactly supported functions to be determined.

After some computations, we have

$$\underline{u}_t=(T-t)^{-(\gamma_1+1)}\left(\gamma_1f_1+\sigma_1\xi_1\frac{df_1}{d\xi_1}\right)$$

$$\left(\underline{u}^{m_1-1}\left|\frac{\partial\underline{u}^{k_1}}{\partial x}\right|^{p_1-2}\frac{\partial\underline{u}}{\partial x}\right)=(T-t)^{-\gamma_1(m_1-1)-(\gamma_1k_1+\sigma_1)(p_1-2)-\gamma_1-\sigma_1}\ f_1^{m_1-1}\left|\frac{df_1^{k_1}}{d\xi_1}\right|^{p_1-2}\frac{df_1}{d\xi_1}$$

$$v^{\alpha_1}\frac{\partial}{\partial x}\left(\underline{u}^{m_1-1}\left|\frac{\partial\underline{u}^{k_1}}{\partial x}\right|^{p_1-2}\frac{\partial\underline{u}}{\partial x}\right)=(T-t)^{-\gamma_1(m_1-1)-(\gamma_1k_1+\sigma_1)(p_1-2)-\gamma_1-2\sigma_1-\alpha_1\gamma_2}\ f_1^{q_1}\cdot$$

$$\cdot\frac{d}{d\xi_1}\left(f_1^{m_1-1}\left|\frac{df_1^{k_1}}{d\xi_1}\right|^{p_1-2}\frac{df_1}{d\xi_1}\right)$$

$$\underline{v}_t=(T-t)^{-(\gamma_2+1)}\left(\gamma_2f_2+\sigma_2\xi_2\frac{df_2}{d\xi_2}\right)$$

$$\left(\underline{v}^{m_2-1}\left|\frac{\partial\underline{v}^{k_2}}{\partial x}\right|^{p_2-2}\frac{\partial\underline{v}}{\partial x}\right)=(T-t)^{-\gamma_2(m_2-1)-(\gamma_2k_2+\sigma_2)(p_2-2)-\gamma_2-\sigma_2}\ f_2^{m_2-1}\left|\frac{df_2^{k_2}}{d\xi_2}\right|^{p_2-2}\frac{df_2}{d\xi_2}$$

$$\underline{u}^{\alpha_2}\frac{\partial}{\partial x}\left(\underline{v}^{m_2-1}\left|\frac{\partial\underline{v}^{k_2}}{\partial x}\right|^{p_2-2}\frac{\partial\underline{v}}{\partial x}\right)=(T-t)^{-\gamma_2(m_2-1)-(\gamma_2k_2+\sigma_2)(p_2-2)-\gamma_2-2\sigma_2-\gamma_1\alpha_2}\ f_1^{q_2}\cdot$$

$$\cdot\frac{d}{d\xi_2}\left(f_2^{m_2-1}\left|\frac{df_2^{k_2}}{d\xi_2}\right|^{p_2-2}\frac{df_2}{d\xi_2}\right)$$

and for the boundary

$$\underline{u}^{m_1-1}\left|\frac{\partial \underline{u}^{k_1}}{\partial x}\right|^{p_1-2}\frac{\partial \underline{u}}{\partial x}\Bigg|_{x=0}=(T-t)^{-\gamma_1(m_1-1)-(\gamma_1 k_1+\sigma_1)(p_1-2)-\gamma_1-\sigma_1}\ f_1^{m_1-1}\left|\frac{df_1^{k_1}}{d\xi_1}\right|^{p_1-2}\frac{df_1}{d\xi_1}(0)$$

$$\underline{v}^{q_1}(0,t)=(T-t)^{-\gamma_2 q_1}\ f_2^{q_1}(0)$$

$$\underline{v}^{m_2-1}\left|\frac{\partial \underline{v}^{k_2}}{\partial x}\right|^{p_2-2}\frac{\partial \underline{v}}{\partial x}\Bigg|_{x=0}=(T-t)^{-\gamma_2(m_2-1)-(\gamma_2 k_2+\sigma_2)(p_2-2)-\gamma_2-\sigma_2}\ f_2^{m_2-1}\left|\frac{df_2^{k_2}}{d\xi_2}\right|^{p_2-2}\frac{df_2}{d\xi_2}(0)$$

$$\underline{u}^{q_2}(0,t)=(T-t)^{-\gamma_1 q_2}\ f_1^{q_2}(0)$$

Notice that

$$\gamma_1+1=\gamma_1(m_1-1)+(\gamma_1 k_1+\sigma_1)(p_1-2)+\gamma_1+2\sigma_1+\alpha_1\gamma_2$$
$$\gamma_2+1=\gamma_2(m_2-1)+(\gamma_2 k_2+\sigma_2)(p_2-2)+\gamma_2+2\sigma_2+\gamma_1\alpha_2$$

$$\gamma_1(m_1-1)+(\gamma_1 k_1+\sigma_1)(p_1-2)+\gamma_1+\sigma_1=\gamma_2 q_1$$
$$\gamma_2(m_2-1)+(\gamma_2 k_2+\sigma_2)(p_2-2)+\gamma_2+\sigma_2=\gamma_1 q_2$$

Thus, ($\underline{u},\underline{v}$) is subsolution of (1)-(3) provided that

$$\begin{gathered} f_2^{\alpha_1}\frac{d}{d\xi_1}\left(f_1^{m_1-1}\left|\frac{df_1^{k_1}}{d\xi_1}\right|^{p_1-2}\frac{df_1}{d\xi_1}\right)\geq\gamma_1 f_1+\sigma_1\xi_1\frac{df_1}{d\xi_1} \\ f_1^{\alpha_2}\frac{d}{d\xi_2}\left(f_2^{m_2-1}\left|\frac{df_2^{k_2}}{d\xi_2}\right|^{p_2-2}\frac{df_2}{d\xi_2}\right)\geq\gamma_2 f_2+\sigma_2\xi_2\frac{df_2}{d\xi_2} \end{gathered}, \qquad (5)$$

$$\begin{gathered} -f_1^{m_1-1}\left|\frac{df_1^{k_1}}{d\xi_1}\right|^{p_1-2}\frac{df_1}{d\xi_1}(0)\leq f_2^{q_1}(0) \\ -f_2^{m_2-1}\left|\frac{df_2^{k_2}}{d\xi_2}\right|^{p_2-2}\frac{df_2}{d\xi_2}(0)\leq f_1^{q_2}(0) \end{gathered}, \qquad (6)$$

Set

$$\begin{gathered} f_1(\xi_1)=A_1(a-\xi)^{c_1} \\ f_2(\xi_2)=A_2(a-\xi)^{c_2} \end{gathered}, \qquad (7)$$

where

$$c_1=\frac{\alpha_1(p_2-1)-(p_1-1)(s_2-1)}{\alpha_1\alpha_2-(s_1-1)(s_2-1)},\ c_2=\frac{\alpha_2(p_1-1)-(p_2-1)(s_1-1)}{\alpha_1\alpha_2-(s_1-1)(s_2-1)},$$

A and a are constants that should be determined. With transformation (7) inequalities (5) become

$$A_1^{s_1}A_2^{\alpha_1}c_1^{p_1-2}k_1^{p_1-2}c_1(c_1-c_2\alpha_1)(a-\xi)^{c_1-1}+A_1c_1\sigma_1\xi_1(a-\xi)^{c_1-1}-\gamma_1A_1(a-\xi)^{c_1}\geq 0$$
$$A_1^{\alpha_2}A_2^{s_2}c_2^{p_2-2}k_2^{p_2-2}c_2(c_2-c_1\alpha_2)(a-\xi)^{c_2-1}+A_2c_2\sigma_2\xi_2(a-\xi)^{c_2-1}-\gamma_2A_2(a-\xi)^{c_2}\geq 0$$

Now we consider the case

$$A_1^{s_1-\alpha_2-1}k_1^{p_1-2}c_1^{p_1-1}\left(c_1-c_2\alpha_1\right)\left(\gamma_2-c_2\sigma_2\right)=A_2^{s_2-\alpha_1-1}k_2^{p_2-2}c_2^{p_2-1}\left(c_2-c_1\alpha_2\right)\left(\gamma_1-c_1\sigma_1\right)$$

and choose

$$a=A_1^{s_1-1}A_2^{\alpha_1}w\,, \tag{8}$$

where

$$w=c_1^{p_1-1}k_1^{p_1-2}\,\frac{c_1-c_2\alpha_1}{\gamma_1-c_1\sigma_1}\,.$$

Here we remark that the assumption

$$\left(p_1q_1+\alpha_1\left(p_1-1\right)\right)\left(p_2q_2+\alpha_2\left(p_2-1\right)\right)>\left(s_1+p_1-1\right)\left(s_2+p_2-1\right)$$

imply $\gamma_1>0,\ \gamma_2>0$, then the inequalities (5) hold.

On the other hand, the boundary conditions in (6) are satisfied if we have

$$\begin{aligned}&A_1^{s_1}\rho_1 a^{c_1-c_2\alpha_1}\le A_2^{q_1}a^{c_2q_1}\\&A_2^{s_2}\rho_2 a^{c_2-c_1\alpha_2}\le A_1^{q_2}a^{c_1q_2}\end{aligned}, \tag{9}$$

where $\rho_1=c_1^{p_1-1}k_1^{p_1-2},\ \rho_2=c_2^{p_2-1}k_2^{p_2-2}$

The condition ensures that we can take A_1 and A_2 large enough such that the inequalities (9) are valid

$$\left(p_1q_1+\alpha_1\left(p_1-1\right)\right)\left(p_2q_2+\alpha_2\left(p_2-1\right)\right)>\left(s_1+p_1-1\right)\left(s_2+p_2-1\right)$$

Therefore, if the initial data $u_0,\ v_0$ is a subsolution to (1)-(3). By the comparison principle, it implies that the solution of (1)-(3) with large initial data blow up in a finite time. The proof is complete.

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