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Parameterization Method for a Fredholm Integro-Differential Equation in Locally Convex Lattice Algebra

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Abstract. *We study a two-point boundary value problem for a Fredholm integro-differential equation with a supremum term in a complete Hausdorff unital locally convex lattice algebra. The topology of the underlying algebra is generated by a directed family of continuous monotone seminorms, which makes it possible to replace norm estimates by seminormwise estimates while retaining the order structure required by the supremum operator. After rewriting the differential equation in variation-of-constants form, we establish a uniform contraction estimate in the locally convex function space $BD([0,T],\mathscr{A}^n)$. Under a contraction bound that is uniform over the defining seminorms, the associated functional-integral equation has a unique solution. To treat the boundary condition constructively, the interval is partitioned and the values of the solution at the left endpoints are introduced as additional parameters. This reduces the original problem to special Cauchy problems on the subintervals together with a block operator equation for the parameters. Topological invertibility of the parameter operator and a uniform contraction condition for the coupled parameter-solution iteration yield unique solvability of the boundary value problem and convergence of the proposed successive approximation scheme in every defining seminorm. The result extends the classical Banach-space parameterization framework to a locally multiplicatively convex lattice-algebra setting; the Banach lattice algebra case is recovered when the defining family consists of a single norm.*

Key words: *Fredholm integro-differential equation, boundary value problem, parameterization method, locally convex lattice algebra, monotone seminorm, contraction mapping, solvability.*

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1 Introduction

Integro-differential equations arise naturally in models that combine local evolution with nonlocal or hereditary effects. Such equations occur, for example, in erosion processes, viscoelasticity, heat conduction with memory, and related problems in applied analysis; see [1]–[4].

Their analytical and numerical treatment has therefore motivated a substantial literature on Fredholm and Fredholm–Volterra integro-differential equations, fixed-point techniques, collocation and semi-analytical methods, inverse and direct boundary value problems, and equations with degenerate kernels [5]–[17].

A particularly useful approach to boundary value problems is the parameterization method developed by Dzhumabaev. Its basic idea is to partition the interval of definition, introduce the values of the unknown solution at selected partition points as additional parameters, and replace the original boundary value problem by a family of special Cauchy problems coupled through algebraic matching and boundary conditions. In the finite-dimensional and Banach-space settings, this construction leads to solvability criteria expressed through a system of algebraic equations for the parameters and also provides a practical iterative procedure for computing the solution. Subsequent developments of the method for nonlocal, hyperbolic, and integro-differential problems are given in [18]–[24].

Equations containing maxima or suprema introduce an additional difficulty because the nonlocal nonlinear term depends not only on the topology of the state space but also on its order structure. Recent works have considered boundary value and functional-differential problems with maxima in classical normed settings [25], [26]. However, many natural function algebras are not conveniently described by a single norm. They are instead modeled by locally convex algebras whose topologies are generated by families of seminorms. This motivates the extension of the parameterization method to a setting in which multiplicative estimates, completeness, and lattice operations can be handled simultaneously.

The purpose of the present paper is to develop such an extension for a Fredholm integro-differential equation with a supremum term in a complete Hausdorff unital locally convex lattice algebra $\mathscr{A}$. The topology of $\mathscr{A}$ is generated by a directed family $\{p_\alpha\}_{\alpha\in\Lambda}$ of continuous monotone seminorms. Passing to $E = \mathscr{A}^n$ and to the function space $BD([0,T],E)$ allows the differential equation, the Fredholm integral term, and the lattice supremum to be estimated seminormwise. A seminorm-Lipschitz estimate for the supremum operator supplies the key control needed for the nonlinear maximum term.

The main contributions of the paper are as follows. First, the boundary value problem is formulated in the locally multiplicatively convex lattice-algebra framework and transformed, by variation of constants, into a functional-integral equation. Second, a uniform seminorm contraction estimate is derived; when the contraction constants are bounded by one common number $q < 1$, the functional-integral equation has a unique solution in $BD([0,T],E)$. Third, a regular partition of $[0,T]$ and additional endpoint parameters are introduced, yielding special Cauchy problems on the subintervals and a block operator equation for the parameter vector. Fourth, topological invertibility of this block operator, together with a uniform contraction condition for the coupled parameter-solution mapping, gives a unique solution of the original boundary value problem. Finally, a successive approximation algorithm is obtained and shown to converge in every defining seminorm. In the special case in which $\mathscr{A}$ is a Banach lattice algebra and the topology is generated by one norm, the results reduce to the familiar Banach-space contraction and parameterization theory.

The paper is organized as follows. Section 2 introduces the locally convex algebraic and lattice setting and the seminorms used on the solution space. Section 3 formulates the boundary value problem and derives its integral representation. Section 4 establishes the seminorm contraction estimate and the corresponding existence and uniqueness theorem. Sections 5 and

6 develop the parameterization on a partition and derive the block operator equation for the parameters. Section 7 presents the computational iteration, and Section 8 states the resulting solvability theorem. Section 9 discusses the scope, interpretation, and limitations of the obtained results, and Section 10 summarizes the main conclusions.

2 Locally convex algebraic setting

Let $\mathscr{A}$ be a complete Hausdorff unital locally convex algebra over $\mathbb{R}$. Its topology is generated by a directed family

$$\mathscr{P} = \{p_\alpha : \alpha \in \Lambda\}$$

of continuous monotone seminorms. We assume that $\mathscr{A}$ is *locally convex lattice spase*, that the lattice operations are continuous.

Set $\mathscr{E} = \mathscr{A}^n$. For $x = (x_1, \ldots, x_n) \in \mathscr{E}$, define

$$p_\alpha^{(n)}(x) = \max_{1 \le i \le n} p_\alpha(x_i). \tag{2.1}$$

Then $\mathscr{E}$ is a complete Hausdorff locally convex space. The componentwise lattice maximum of two vectors is denoted by $x \vee y$.

For a continuous function $x : [0,T] \to \mathscr{E}$, put

$$p_{\alpha,\infty}(x) = \sup_{t \in [0,T]} p_\alpha^{(n)}(x(t)). \tag{2.2}$$

For $x \in C^1([0,T], \mathscr{E})$ and a fixed number $h > 0$, define

$$q_{\alpha,h}(x) = p_{\alpha,\infty}(x) + h\, p_{\alpha,\infty}(x'). \tag{2.3}$$

The family $\{q_{\alpha,h}\}_{\alpha \in \Lambda}$ generates a complete locally convex topology on $BD([0,T], \mathscr{E}) := C^1([0,T], \mathscr{E})$.

For a continuous function $x : [0,T] \to \mathscr{E}$ and a compact interval $\Omega \subset [0,T]$, define the componentwise supremum

$$\sup\bigl\{x(\tau) : \tau \in \Omega\bigr\} = \left(\sup\bigl\{x_1(\tau) : \tau \in \Omega\bigr\}, \ldots, \sup\bigl\{x_n(\tau) : \tau \in \Omega\bigr\}\right), \tag{2.4}$$

whenever these suprema exist in $\mathscr{A}$. We assume throughout that the relevant order intervals are order complete, so that (2.4) is well defined.

We assume that the maximum analog of the Lemma in [27] is true for the following suremum case.

Lemma 2.1 (Seminorm-Lipschitz property of the supremum)**.** *It is true there for every $\alpha \in \Lambda$, every $x, y \in BD([0,T], \mathscr{E})$, and all $t \in [0,T]$ that*

$$p_\alpha^{(n)} \left(\sup\bigl\{x(\tau) : \tau \in \Omega(t)\bigr\} - \sup\bigl\{y(\tau) : \tau \in \Omega(t)\bigr\}\right) \le$$
$$\le p_{\alpha,\infty}(x - y) + h\, p_{\alpha,\infty}(x' - y'), \tag{2.5}$$

where $\Omega(t) = \bigl[\delta_1(t); \delta_2(t)\bigr],\ \delta_1(t), \delta_2(t) \in C[0,T],\ 0 \le \delta_1(t) \le \delta_2(t) \le t,\ 0 < h = \sup\limits_{0 \le t \le T} \bigl|\, \delta_1(t) \bigr| + \sup\limits_{0 \le t \le T} \bigl|\, \delta_2(t) \bigr|.$

Remark 2.1. Lemma 2.1 is automatic in standard coordinatewise ordered function algebras such as $C(K)^n$ equipped with monotone seminorms obtained from supremum over compact subsets.

3 Problem statement

In the present paper, on the interval $[0, T]$ we consider the "linear" integro-differential equation

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)\sup\big\{x(\tau) : \tau \in \Omega(t)\big\} + \int\limits_0^T K(t,s)x(s)ds + f(t), \quad t \in (0,T), \tag{3.1}$$

with boundary conditions

$$B_0x(0) + C_0x(T) = D_0, \tag{3.2}$$

where
$A, B \in C([0,T], \mathscr{L}(\mathscr{E}))$; $\quad K \in C([0,T]^2, \mathscr{L}(\mathscr{E}))$; $\quad f \in C([0,T], \mathscr{E})$;
B_0, $C_0 \in \mathscr{L}(\mathscr{E})$ and $D_0 \in \mathscr{E}$; $\quad \Omega(t) = \big[\delta_1(t); \delta_2(t)\big]$.

All integrals are Riemann integrals in the complete locally convex space $\mathscr{E}$, equivalently seminormwise limits of Riemann sums. For a continuous linear operator $L \in \mathscr{L}(\mathscr{E})$, let

$$\|L\|_\alpha = \sup\{p_\alpha^{(n)}(Lx) : p_\alpha^{(n)}(x) \le 1\}, \tag{3.3}$$

whenever L is p_α-bounded. We assume that all coefficient operators are bounded with respect to each defining seminorm.

A solution of (3.1), (3.2) is a function $x \in BD([0,T], \mathscr{E})$ satisfying the differential equation and the boundary condition.

Let $X(t)$ be the evolution operator of

$$X'(t) = A(t)X(t), \qquad X(0) = I. \tag{3.4}$$

We assume that $X(t)$ is invertible for all t and that X, X^{-1} are continuous in every operator seminorm. By variation of constants, every solution of (3.1) satisfies

$$x(t) = X(t)x(0) + X(t)\int\limits_0^t X^{-1}(s)B(s)\sup\big\{x(\tau) : \tau \in \Omega(s)\big\}ds +$$

$$+X(t)\int\limits_0^t X^{-1}(s)\int\limits_0^T K(s,\theta)x(\theta)\,d\theta ds + X(t)\int\limits_0^t X^{-1}(s)f(s)ds, \tag{3.5}$$

where $\Omega(t) = \big[\delta_1(t); \delta_2(t)\big]$.

4 Existence and uniqueness of solution of the equation (3.5)

We use the formula (2.3), where $0 < h = \sup\limits_{0\le t\le T}\big|\delta_1(t)\big| + \sup\limits_{0\le t\le T}\big|\delta_2(t)\big|$. In our further calculations we use the inequality (2.5). We calculate the derivative from the equation

$$x'(t) = X'(t)x(0) + X'(t)\int\limits_0^t X^{-1}(s)B(s)\sup\big\{x(\tau) : \tau \in \Omega(s)\big\}ds +$$

$$+X(t)X^{-1}(t)B(t)\sup\big\{x(\tau) : \tau \in \Omega(t)\big\} +$$

$$+X'(t)\int\limits_0^t X^{-1}(s)\int\limits_0^T K(s,\theta)x(\theta)d\theta ds + X(t)X^{-1}(t)\int\limits_0^T K(t,s)x(s)ds+$$

$$+X'(t)\int\limits_0^t X^{-1}(s)f(s)ds + X(t)X^{-1}(t)f(t). \tag{4.1}$$

We denote

$$g(t) = X(t)x(0) + X(t)\int\limits_0^t X^{-1}(s)f(s)ds,$$

$$g'(t) = X'(t)x(0) + X'(t)\int\limits_0^t X^{-1}(s)f(s)ds + X(t)X^{-1}(t)f(t).$$

Define the operator $\mathbb{T}$ on $BD([0,T],\mathscr{E})$ by the right-hand side of (3.5), with a fixed initial value $x(0) = x_0$. For each $\alpha \in \Lambda$, put

$$\begin{gathered} M_{X,\alpha} = \sup_{t\in[0,T]} \| X(t) \|_\alpha, \quad M_{X^{-1},\alpha} = \sup_{t\in[0,T]} \| X^{-1}(t) \|_\alpha, \qquad M_{B,\alpha} = \sup_{t\in[0,T]} \| B(t) \|_\alpha, \\ M_{f,\alpha} = \sup_{t\in[0,T]} \| f(t) \|_\alpha, \qquad M_{K,\alpha} = \sup_{t\in[0,T]} \int\limits_0^T \| K(t,s) \|_\alpha \, ds. \end{gathered} \tag{4.2}$$

Lemma 4.1. *Under the assumption of the Lemma 2.1, for every $x, y \in BD([0,T],\mathscr{E})$ and every $\alpha \in \Lambda$, there holds*

$$q_{\alpha,h}(\mathbb{T}x - \mathbb{T}y) \le \rho_\alpha q_{\alpha,h}(x-y), \tag{4.3}$$

where $\rho_\alpha = \sup_{\alpha\in\Lambda}\Big\{C_{0\alpha} + h\,C_{2\alpha}\Big\} < 1$, $C_{0\alpha}$ and $C_{2\alpha}$ are defined from (4.7) *and* (4.12) *below.*

Proof. Taking (3.3) and (4.2) into account, for our denatations we obtain that the following estimates are true:

$$p_\alpha^{(n)}\big(g(t)\big) \le p_\alpha^{(n)}\big(X(t)\big)p_\alpha^{(n)}\big(x(0)\big) + p_\alpha^{(n)}\big(X(t)\big)\int\limits_0^t p_\alpha^{(n)}\big(X^{-1}(s)\big)p_\alpha^{(n)}\big(f(s)\big)ds \le$$

$$\le \sup_{t\in[0,T]} \| X(t) \|_\alpha \big\| x(0) \big\|_\alpha + \sup_{t\in[0,T]} \| X(t) \|_\alpha \int\limits_0^t \sup_{s\in[0,T]} \| X^{-1}(s) \|_\alpha \sup_{s\in[0,T]} \| f(s) \|_\alpha ds \le$$

$$\le M_{X,\alpha}\big\| x(0) \big\|_\alpha + M_{X,\alpha}TM_{X^{-1},\alpha}M_{f,\alpha} = g_{0\alpha} = \text{const} < \infty, \tag{4.4}$$

$$p_\alpha^{(n)}\big(g'(t)\big) \le p_\alpha^{(n)}\big(X'(t)\big)p_\alpha^{(n)}\big(x(0)\big) + p_\alpha^{(n)}\big(X'(t)\big)\int\limits_0^t p_\alpha^{(n)}\big(X^{-1}(s)\big)p_\alpha^{(n)}\big(f(s)\big)ds+$$

$$+p_\alpha^{(n)}\big(X(t)\big)p_\alpha^{(n)}\big(X^{-1}(t)\big)p_\alpha^{(n)}\big(f(t)\big) \le$$

$$\le p_{\alpha,\infty}\big(X'(t)\big)p_\alpha\big(x(0)\big) + Tp_{\alpha,\infty}\big(X'(t)\big)p_{\alpha,\infty}\big(X^{-1}(t)\big)p_{\alpha,\infty}\big(f(t)\big)+$$

$$+p_{\alpha,\infty}\big(X(t)\big)p_{\alpha,\infty}\big(X^{-1}(t)\big)p_{\alpha,\infty}\big(f(t)\big) \leq$$

$$\leq M_{X',\alpha}\big\| x(0) \big\|_\alpha + TM_{X',\alpha}M_{X^{-1},\alpha}M_{f,\alpha} + M_{X,\alpha}M_{X^{-1},\alpha}M_{f,\alpha} < g'_{0\alpha} = \text{const} < \infty. \tag{4.5}$$

For arbitrary difference of the operator $\mathbb{T}$ we obtain

$$p_\alpha^{(n)}\big(\mathbb{T}x(t) - \mathbb{T}y(t)\big) \leq p_\alpha^{(n)}\big(X(t)\big)\int\limits_0^t p_\alpha^{(n)}\big(X^{-1}(s)\big)\Big[p_\alpha^{(n)}\big(B(s)\big)p_\alpha^{(n)}\big(\sup\big\{x(\tau) : \tau \in \Omega(s)\big\} -$$

$$-\sup\big\{y(\tau) : \tau \in \Omega(s)\big\}\big) + \int\limits_0^T p_\alpha^{(n)}\big(K(s,\theta)\big)p_\alpha^{(n)}\big(x(\theta) - y(\theta)\big)d\theta\Big]ds \leq$$

$$\leq \Big[Tp_{\alpha,\infty}\big(X(t)\big)p_{\alpha,\infty}\big(X^{-1}(t)\big) + p_{\alpha,\infty}\big(K(t,s)\big)\Big]p_{\alpha,\infty}\big(x(t) - y(t)\big) +$$

$$+Tp_{\alpha,\infty}\big(X(t)\big)p_{\alpha,\infty}\big(X^{-1}(t)\big)p_\alpha^{(n)}\big(B(t)\big)p_{\alpha,\infty}\big(\sup\big\{x(\tau) : \tau \in \Omega(t)\big\} - \sup\big\{y(\tau) : \tau \in \Omega(t)\big\}\big).$$

Applying the inequalities (2.5) in Lemma 2.1 to the last estimate, we obtain

$$p_{\alpha,\infty}\big(\mathbb{T}x(t) - \mathbb{T}y(t)\big) \leq C_{0\alpha}p_{\alpha,\infty}\big(x(t) - y(t)\big) + h\,C_{1\alpha}p_{\alpha,\infty}\big(x'(t) - y'(t)\big), \tag{4.6}$$

where

$$C_{0\alpha} = TM_{X,\alpha}M_{X^{-1},\alpha}\big[M_{B,\alpha} + M_{K,\alpha}\big], \quad C_{1\alpha} = TM_{X,\alpha}M_{X^{-1},\alpha}M_{B,\alpha}. \tag{4.7}$$

Since $C_{0\alpha} \geq C_{1\alpha}$, then from (4.6) we have

$$p_{\alpha,\infty}\big(\mathbb{T}x(t) - \mathbb{T}y(t)\big) \leq C_{0\alpha}\Big[p_{\alpha,\infty}\big(x(t) - y(t)\big) + h\,p_{\alpha,\infty}\big(x'(t) - y'(t)\big)\Big]. \tag{4.8}$$

Similarly, from the equation (4.1) we derive

$$p_\alpha^{(n)}\big(\mathbb{T}x'(t) - \mathbb{T}y'(t)\big) \leq$$

$$\leq p_\alpha^{(n)}\big(X'(t)\big)\int\limits_0^t p_\alpha^{(n)}\big(X^{-1}(s)\big)p_\alpha^{(n)}\big(B(s)\big)p_\alpha^{(n)}\big(\sup\big\{x(\tau) : \tau \in \Omega(s)\big\} - \sup\big\{y(\tau) : \tau \in \Omega(s)\big\}\big)ds +$$

$$+p_\alpha^{(n)}\big(X(t)\big)p_\alpha^{(n)}\big(X^{-1}(t)\big)(t)p_\alpha^{(n)}\big(B(t)\big)p_\alpha^{(n)}\big(\sup\big\{x(\tau) : \tau \in \Omega(t)\big\} - \sup\big\{y(\tau) : \tau \in \Omega(t)\big\}\big) +$$

$$+p_\alpha^{(n)}\big(X'(t)\big)\int\limits_0^t p_\alpha^{(n)}\big(X^{-1}(s)\big)\int\limits_0^T p_\alpha^{(n)}\big(K(s,\theta)\big)p_\alpha^{(n)}\big(x(\theta) - y(\theta)\big)d\theta ds +$$

$$+p_\alpha^{(n)}\big(X(t)\big)p_\alpha^{(n)}\big(X^{-1}(t)\big)\int\limits_0^T p_\alpha^{(n)}\big(K(t,s)\big)p_\alpha^{(n)}\big(x(s) - y(s)\big)ds \leq$$

$$\leq T\sup_{t\in[0,T]} \| X'(t) \|_\alpha \sup_{t\in[0,T]} \| X^{-1}(t) \|_\alpha \sup_{t\in[0,T]} \| B(t) \|_\alpha \times$$

$$\times \sup_{t\in[0,T]} \| \sup\big\{x(\tau) : \tau \in \Omega(t)\big\} - \sup\big\{y(\tau) : \tau \in \Omega(t)\big\} \|_\alpha +$$

$$+\sup_{t\in[0,T]} \| X(t) \|_\alpha \sup_{t\in[0,T]} \| X^{-1}(t) \|_\alpha \sup_{t\in[0,T]} \| B(t) \|_\alpha \times$$

$$\times \sup_{t\in[0,T]} \| \sup\big\{x(\tau):\tau\in\Omega(t)\big\} - \sup\big\{x(\tau):\tau\in\Omega(t)\big\} \|_\alpha +$$

$$+T\sup_{t\in[0,T]}\| X'(t)\|_\alpha \sup_{t\in[0,T]}\| X^{-1}(t)\|_\alpha \sup_{t\in[0,T]}\| K(t,s)\|_\alpha \sup_{t\in[0,T]}\| x(t)-y(t)\|_\alpha +$$

$$+\sup_{t\in[0,T]}\| X(t)\|_\alpha \sup_{t\in[0,T]}\| X^{-1}(t)\|_\alpha \sup_{t\in[0,T]}\| K(t,s)\|_\alpha \sup_{t\in[0,T]}\| x(t)-y(t)\|_\alpha. \tag{4.9}$$

The inequality (4.9) we rewrite as

$$p_{\alpha,\infty}\big(\mathbb{T}x'(t)-\mathbb{T}y'(t)\big) \le$$

$$\le TM_{X',\alpha}M_{X^{-1},\alpha}M_{B,\alpha}p_{\alpha,\infty}\big(\sup\big\{x(\tau):\tau\in\Omega(t)\big\} - \sup\big\{y(\tau):\tau\in\Omega(t)\big\}\big)+$$

$$+M_{X,\alpha}M_{X^{-1},\alpha}M_{B,\alpha}p_{\alpha,\infty}\big(\sup\big\{x(\tau):\tau\in\Omega(t)\big\} - \sup\big\{x(\tau):\tau\in\Omega(t)\big\}\big)+$$

$$+TM_{X',\alpha}M_{X^{-1},\alpha}M_{K,\alpha}p_{\alpha,\infty}\big(x(t)-y(t)\big)+$$

$$+M_{X,\alpha}M_{X^{-1},\alpha}M_{K,\alpha}p_{\alpha,\infty}\big(x(t)-y(t)\big) \le$$

$$\le M_{X^{-1},\alpha}M_{B,\alpha}\Big(TM_{X',\alpha}+M_{X,\alpha}\Big)p_{\alpha,\infty}\big(\sup\big\{x(\tau):\tau\in\Omega(t)\big\} - \sup\big\{x(\tau):\tau\in\Omega(t)\big\}\big)+$$

$$+M_{X^{-1},\alpha}M_{K,\alpha}\Big(TM_{X',\alpha}+TM_{X,\alpha}\Big). \tag{4.10}$$

Applying the inequalities (2.5) in Lemma 2.1, (4.4) and (4.5) to the last estimate (4.10), we get

$$p_{\alpha,\infty}\big(\mathbb{T}x'(t)-\mathbb{T}y'(t)\big) \le C_{2\alpha}p_{\alpha,\infty}\big(x(t)-y(t)\big) + h\,C_{3\alpha}p_{\alpha,\infty}\big(x'(t)-y'(t)\big), \tag{4.11}$$

where

$$C_{2\alpha} = M_{X^{-1},\alpha}\big(TM_{X',\alpha}+M_{X,\alpha}\big)\big(M_{B,\alpha}+M_{K,\alpha}\big),\ \ C_{3\alpha} = M_{X^{-1},\alpha}M_{B,\alpha}\big(TM_{X',\alpha}+M_{X,\alpha}\big). \tag{4.12}$$

Since $C_{2\alpha}\ge C_{3\alpha}$, then from (4.11) we have

$$p_{\alpha,\infty}\big(\mathbb{T}x'(t)-\mathbb{T}y'(t)\big) \le C_{2\alpha}\Big[p_{\alpha,\infty}\big(x(t)-y(t)\big) + h\,p_{\alpha,\infty}\big(x'(t)-y'(t)\big)\Big]. \tag{4.13}$$

According to the (2.3), from estimates (4.8) and (4.13) we get the estimate (4.3)

$$q_{\alpha,h}(\mathbb{T}x(t)-\mathbb{T}y(t)) \le \rho_\alpha\Big[p_{\alpha,\infty}\big(x(t)-y(t)\big) + h\,p_{\alpha,\infty}\big(x'(t)-y'(t)\big)\Big] = \rho_\alpha q_{\alpha,h}\big(x(t)-y(t)\big),$$

where $\rho_\alpha = C_{0\alpha}+h\,C_{2\alpha}$, $C_{0\alpha}$ and $C_{2\alpha}$ are defined from (4.7) and (4.12), respectively.

The fact that $\rho_\alpha = \sup_{\alpha\in\Lambda}\Big\{C_{0\alpha}+h\,C_{2\alpha}\Big\} < 1$, is finished the proof of the Lemma 4.1. □

Theorem 4.1 (Uniform seminorm contraction)**.** *Assume that:*
$\mathscr{A}$ is a complete Hausdorff unital locally m-convex lattice algebra;
Lemma 4.1 holds and all coefficient operators are bounded with respect to every p_α;
there exists a number $q\in(0,1)$ such that

$$\sup_{\alpha\in\Lambda}\rho_\alpha \le q < 1. \tag{4.14}$$

Then, for every prescribed $x_0\in\mathscr{E}$, the functional-integral equation (3.5) *has a unique solution in $BD([0,T],\mathscr{E})$.*

Proof. Choose $x^{(0)} \in BD\big([0,T],\mathscr{E}\big)$ and define $x^{(k+1)} = \mathbb{T}x^{(k)}$. By Lemma 4.1,

$$q_{\alpha,h}(x^{(k+1)} - x^{(k)}) \le q^k q_{\alpha,h}(x^{(1)} - x^{(0)})$$

for every α. Hence $(x^{(k)})$ is Cauchy for every defining seminorm. Completeness gives a limit $x \in BD([0,T],\mathscr{E})$. Continuity of $\mathbb{T}$ implies $\mathbb{T}x = x$. If x and y are fixed points, then

$$q_{\alpha,h}(x-y) \le q q_{\alpha,h}(x-y),$$

so $q_{\alpha,h}(x-y) = 0$ for every α. Since the topology is Hausdorff, $x = y$. $\square$

Remark 4.1. A weaker, seminorm-dependent condition can be formulated by using a calibrated family of seminorms and a generalized contraction theorem for locally convex spaces. Condition (4.14) is stronger but yields a direct and transparent extension of the Banach contraction argument.

5 Parameterization on a partition

Let $N \in \mathbb{N}$, $h_0 = T/N$, and

$$t_r = rh_0, \qquad r = 0,1,\dots,N.$$

On each interval $[t_{r-1}, t_r)$ introduce

$$x_r(t) = u_r(t) + \lambda_r, \qquad \lambda_r = x_r(t_{r-1}), \qquad u_r(t_{r-1}) = 0.$$

Here $\lambda = (\lambda_1,\dots,\lambda_N) \in \mathscr{E}^N$ is the parameter vector. The differential equation becomes

$$\frac{du_r(t)}{dt} = A(t)\big(u_r(t)+\lambda_r\big) + B(t)\sup\Big\{\big(u_r(\tau)+\lambda_r\big) : \tau \in \Omega(t)\Big\} +$$

$$+\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} K(t,s)\big(u_j(s)+\lambda_j\big)ds + f(t), \;\; t \in \big[t_{r-1},t_r\big), \;\; r = \overline{1,N}, \tag{5.1}$$

together with

$$u_r(t_{r-1}) = 0, \qquad r = 1,\dots,N. \tag{5.2}$$

The boundary condition is

$$B_0\lambda_1 + C_0\big(u_N(T^-) + \lambda_N\big) = D_0, \tag{5.3}$$

and the matching conditions are

$$u_r(t_r^-) + \lambda_r = \lambda_{r+1}, \qquad r = 1,\dots,N-1. \tag{5.4}$$

For fixed λ, equations (5.1)–(5.2) form a special Cauchy problem in the locally convex algebra. By Theorem 4.1, applied on each subinterval, this problem has a unique solution whenever the corresponding local contraction constants satisfy the uniform bound.

The problem (5.1)–(5.2) we write as the equivalent system of integral equations

$$u_r(t) = X(t)\int\limits_{(r-1)h_0}^{t} X^{-1}(\theta_1)\big[A(\theta_1)+B(\theta_1)\big]d\theta_1\lambda_r + X(t)\int\limits_{(r-1)h_0}^{t} X^{-1}(\theta_1)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j} K(\theta_1,s)dsd\theta_1\lambda_j +$$

$$+X(t)\int\limits_{(r-1)h_0}^{t} X^{-1}(\theta_1)f(\theta_1)d\theta_1 + X(t)\int\limits_{(r-1)h_0}^{t} X^{-1}(\theta_1)B(\theta_1)\sup\Big\{u_r(\tau):\tau\in\Omega(\theta_1)\Big\}d\theta_1 +$$

$$+X(t)\int\limits_{(r-1)h_0}^{t} X^{-1}(\theta_1)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j} K(\theta_1,s)u_j(\theta_1)dsd\theta_1,\ \ t\in\big[(r-1)h_0,rh_0\big),\ \ r=\overline{1,N}. \qquad (5.5)$$

5.1 Successive-substitution operators

Let $P:[t_{r-1},t_r]\to\mathscr{L}(\mathscr{E})$ be continuous. We suppose that $P(t)$ be an arbitrary square matrix continuous on the interval $\big[(r-1)h_0,rh_0\big)$ and it has a finite limit $\lim\limits_{t\to rh_0-0}P(t),\ r=\overline{1,N}$. Take a $\nu\in\mathbb{N}$ and denote by $\Phi_{\nu,r}\big(A(\cdot),P(\cdot),t\big)$ the sum

$$\int\limits_{(r-1)h_0}^{t} P(s_1)ds_1 + \int\limits_{(r-1)h_0}^{t} A(s_1)\int\limits_{(r-1)h_0}^{s_2} P(s_2)ds_2ds_1 + \ldots +$$

$$+\int\limits_{(r-1)h_0}^{t} A(s_1)\ldots\int\limits_{(r-1)h_0}^{s_{\nu-2}} A(s_{\nu-1})\int\limits_{(r-1)h}^{s_{\nu-1}} P(s_\nu)ds_\nu ds_{\nu-1}\ldots ds_1,\ \ t\in\big[(r-1)h_0,rh_0\big),\ \ r=\overline{1,N}.$$

The sum $\Phi_{\nu,r}(A(\cdot),P(\cdot),t)$ is continuous on $\big[(r-1)h_0,rh_0\big)$ and has a finite limit

$$\lim_{t\to rh_0-0}\Phi_{\nu,r}\big(A(\cdot),P(\cdot),t\big)=\Phi_{\nu,r}\big(A(\cdot),P(\cdot),rh_0\big)\ \text{for all}\ \nu\in\mathbb{N},\ \ r=\overline{1,N}.$$

It is evident that $\Phi_{*,r}\big(A(\cdot),P(\cdot),t\big)=\lim\limits_{\nu\to\infty}\Phi_{\nu,r}(A(\cdot),P(\cdot),t)$ is a sum of uniformly convergent series on $\big[(r-1)h_0,rh_0\big)$, and the sum is continuous on the interval $\big[(r-1)h_0,rh_0\big)$ and has a finite limit

$$\lim_{t\to rh_0-0}\Phi_{*,r}\big(A(\cdot),P(\cdot),t\big)=\Phi_{*,r}\big(A(\cdot),P(\cdot),rh_0\big),\quad r=\overline{1,N}.$$

Substituting the right-hand side of (5.5) into $u_r(s)$ in (5.5) and repeating the process ν ($\nu\in\mathbb{N}$) times, we obtain the following presentation for a function $u_r(t)$:

$$u_r(t)=F_{\nu,r}(t)\lambda_r+\sum_{j=1}^{N}J_{\nu,rj}(t)\lambda_j+H_{\nu,r}(u_r,t)+$$

$$+\Psi_{\nu,r}(u_r,t)+K_{\nu,r}(t),\ \ t\in\big[(r-1)h_0,rh_0\big),\ \ r=\overline{1,N}, \qquad (5.6)$$

where

$$F_{\nu,r}(t)=\Phi_{\nu,r}\big(A(\cdot),A(\cdot)+B(\cdot),t\big),\quad K_{\nu,r}(t)=\Phi_{\nu,r}\big(A(\cdot),f(\cdot),t\big),$$

$$H_{\nu,r}(t) = \Phi_{\nu,r}\big(A(\cdot), B(\cdot)\sup\{u_r(\tau)\}, t\big),$$

and

$$\Psi_{\nu,r}(u_r, t) = \int\limits_{(r-1)h_0}^{t} A(s_1)\ldots \int\limits_{(r-1)h_0}^{s_{\nu-1}} \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} K(\theta, s_\nu)u_j(\theta)\theta ds_\nu \ldots ds_1,$$

$t \in \big[(r-1)h_0, rh_0\big), \quad r = \overline{1, N}.$

$J_{\nu,rj}$ collects the parameter contribution generated by the Fredholm kernel. The terms $H_{\nu,r}$, and $\Psi_{\nu,r}$ contain, respectively, the remaining differential, supremum, and Fredholm contributions involving u. Each term is defined by iterated integrals in $\mathscr{E}$ and is continuous with respect to every defining seminorm.

6 Algebraic system for the parameters

Substituting the endpoint values obtained from (5.6) into (5.3) and (5.4) gives

$$Q_\nu(h_0)\lambda = -b_\nu(h_0) - \mathscr{R}_\nu(u; h_0), \qquad \lambda \in \mathscr{E}^N. \tag{6.1}$$

Here $Q_\nu(h_0)$ is an $N \times N$ block operator matrix over $\mathscr{L}(\mathscr{E})$ of the form

$$Q_\nu(h_0) = \Big[Q_\nu^{(rj)}(h_0)\Big]_{r,j=1}^{N}. \tag{6.2}$$

The operators $\widetilde{J}_{\nu,r}$ represent the endpoint contributions of the Fredholm parameter terms. The vector $b_\nu(h_0)$ contains D_0 and the inhomogeneous terms $K_{\nu,r}(t_r)$, whereas $\mathscr{R}_\nu(u; h_0)$ contains the nonlinear supremum contribution and the remaining $u-$dependent terms.

Definition 6.1. The block operator $Q_\nu(h_0) : \mathscr{E}^N \to \mathscr{E}^N$ is called *topologically invertible* if it is bijective and $Q_\nu(h_0)^{-1}$ is continuous for the product locally convex topology.

Theorem 6.1 (Solvability of the parameter problem)**.** *Suppose that:*
the assumptions of Theorem 4.1 hold on every subinterval $[t_{r-1}, t_r]$ *and* $\delta_1(t) \geq t_{r-1}, \quad t \in [t_{r-1}, t_r]$*;*
for some $\nu \in \mathbb{N}$*, the operator* $Q_\nu(h_0)$ *is topologically invertible;*
for every $\alpha \in \Lambda$ *there is* $c_\alpha > 0$ *such that*

$$p_\alpha^{(nN)}\big(Q_\nu(h_0)^{-1}z\big) \leq c_\alpha p_\alpha^{(nN)}(z), \qquad z \in \mathscr{E}^N; \tag{6.3}$$

the combined parameter-solution iteration has a uniform contraction factor smaller than one with respect to the product family of seminorms.

Then the multipoint problem (5.1)–(5.4) *has a unique solution* (λ, u)*, and the function*

$$x(t) = u_r(t) + \lambda_r, \qquad t \in [t_{r-1}, t_r)$$

is the unique solution of the original boundary value problem (3.1)–(3.2)*.*

Proof. For a fixed parameter vector λ, Theorem 4.1 gives a unique local solution $u(\lambda)$. Equation (6.1) and topological invertibility of $Q_\nu(h_0)$ define

$$\lambda = -Q_\nu(h_0)^{-1}\big[b_\nu(h_0) + \mathscr{R}_\nu(u(\lambda); h_0)\big].$$

By the assumed uniform seminorm contraction property, this mapping has a unique fixed point in $\mathscr{E}^N$. The matching conditions then produce a globally continuous function, and the differential equation holds on every subinterval. Conversely, any solution of the original problem generates parameters and local functions satisfying the parameter problem. Thus the two problems are equivalent. $\square$

7 Computational iteration

Assume that $Q_\nu(h_0)$ is topologically invertible. A practical iteration is as follows.

1. Choose $u^{(0)} = 0$ and compute

$$\lambda^{(0)} = -Q_\nu(h_0)^{-1} b_\nu(h_0). \tag{7.1}$$

2. For $r = 1, \dots, N$, compute $u_r^{(0)}$ from the truncated representation

$$u_r^{(0)}(t) = F_{\nu,r}(t)\lambda_r^{(0)} + \sum_{j=1}^{N} J_{\nu,rj}(t)\lambda_j^{(0)} + K_{\nu,r}(t). \tag{7.2}$$

3. Given $u^{(k)}$, solve

$$\lambda^{(k+1)} = -Q_\nu(h_0)^{-1}\big[b_\nu(h_0) + \mathscr{R}_\nu(u^{(k)}; h_0)\big]. \tag{7.3}$$

4. Update

$$\begin{aligned} u_r^{(k+1)}(t) = F_{\nu,r}(t)\lambda_r^{(k+1)} + \sum_{j=1}^{N} J_{\nu,rj}(t)\lambda_j^{(k+1)} \\ + H_{\nu,r}(u^{(k)}; t) + \Psi_{\nu,r}(u^{(k)}; t) + K_{\nu,r}(t). \end{aligned} \tag{7.4}$$

Under the contraction assumptions of Theorem 6.1, $(\lambda^{(k)}, u^{(k)})$ converges in every defining seminorm to the unique solution (λ, u).

8 Main result

Theorem 8.1. *Let $\mathscr{A}$ be a complete Hausdorff unital locally convex lattice algebra whose topology is generated by a directed family of continuous monotone seminorms. Assume that the supremum operator satisfies* (2.5)*, the coefficient operators are seminorm-bounded, and the uniform contraction condition* (4.14) *holds. Let $h_0 = T/N$ for some $N \in \mathbb{N}$. If there exists $\nu \in \mathbb{N}$ such that the block operator $Q_\nu(h_0)$ is topologically invertible and the coupled parameter iteration is uniformly contractive, then the boundary value problem* (3.1)*–*(3.2) *has a unique solution in $BD([0,T], \mathscr{E})$. Moreover, the iterates* (7.1)*–*(7.4) *converge to that solution in the topology generated by the seminorms $q_{\alpha,h}$.*

Corollary 8.1. *If $\mathscr{A}$ is a Banach lattice algebra and the defining family consists of one norm, Theorem 8.1 reduces to the usual Banach-space contraction and parameterization result.*

9 Discussion

The principal feature of the obtained results is that the parameterization method does not require the state space to be controlled by a single norm. Instead, the analysis is carried out with respect to the directed family of submultiplicative seminorms that generates the topology of the locally convex lattice algebra. This is more than a formal replacement of a norm by a family of seminorms. Existence, uniqueness, continuity of the integral operators, and convergence of the approximation scheme must all be verified simultaneously in the locally convex topology. The uniform estimate

$$\sup_{\alpha\in\Lambda} \rho_\alpha \leq q < 1$$

provides a transparent way to achieve this: the Picard-type iteration is Cauchy in every defining seminorm with a common geometric rate, and completeness then yields a limit in the original function space.

The lattice assumption plays an equally essential role. A supremum term such as

$$\sup\{x(\tau) : \tau \in \Omega\}$$

has no intrinsic meaning in an arbitrary locally convex algebra. The vector-lattice structure and the assumed order completeness of the relevant intervals make this term well defined, while the seminorm-Lipschitz property of the supremum gives the estimate needed to incorporate it into the contraction argument. Thus the algebraic, topological, and order structures are all used in an essential way: local convexity controls products, completeness controls limits of iterations and Riemann sums, and the lattice structure controls the maximum-type nonlinearity.

The parameterization step separates two sources of difficulty in the original boundary value problem. For fixed endpoint parameters, the equation on each subinterval is treated as a special Cauchy problem and is controlled by the local contraction estimates. The global boundary and matching conditions are then transferred to the block operator $Q_\nu(h_0)$. Consequently, unique solvability is governed by two complementary mechanisms: local well-posedness of the subinterval problems and global topological invertibility of the parameter operator. This decomposition is particularly useful computationally because the differential-integral part and the algebraic parameter update can be handled in alternating stages.

Theorem 8.1 therefore provides both an analytical solvability criterion and a constructive approximation procedure. Under the stated assumptions, the iterates generated by the parameter update and the truncated successive-substitution representation converge in the topology generated by the seminorms $q_{\alpha,h}$. The resulting convergence is stronger than convergence with respect to any single selected seminorm: it holds simultaneously for the entire defining family. At the same time, Corollary 8.1 shows that the present framework is consistent with the classical theory, since a Banach lattice algebra is recovered by taking a single submultiplicative norm.

Several assumptions in the present formulation are intentionally strong. In particular, the common contraction bound $\sup_\alpha \rho_\alpha < 1$ is sufficient for a direct Banach-type fixed-point argument but is not expected to be necessary in a general locally convex space. As noted in Remark 4.1, weaker seminorm-dependent conditions may be possible through calibrated families of seminorms or more general fixed-point theorems. Likewise, topological invertibility of $Q_\nu(h_0)$ and uniform contractivity of the coupled parameter-solution iteration are imposed as explicit hypotheses rather than characterized in terms of readily verifiable spectral or coefficient conditions. Developing such criteria would strengthen the practical applicability of the method.

A further direction concerns quantitative computation. The present analysis proves convergence of the successive approximation scheme in every defining seminorm, but it does not derive explicit a posteriori error estimates, optimal choices of the partition size h_0, or truncation rules for ν. Such estimates would be useful for numerical implementation. It would also be natural to investigate whether the theory can be extended to nonuniform partitions, more general delay or memory intervals, nonlinear coefficient operators, and other classes of ordered locally convex lattice algebras. These questions lie beyond the scope of the present paper but follow naturally from the seminormwise parameterization framework developed here.

10 Conclusion

A parameterization method has been developed for a two-point boundary value problem for a Fredholm integro-differential equation with a supremum term in a complete Hausdorff unital locally convex lattice algebra. The locally convex topology is handled through a directed family of continuous monotone seminorms, while the lattice structure provides a meaningful and controllable supremum operator. This setting permits the differential equation to be transformed into a functional-integral equation and analyzed by a uniform seminorm contraction argument.

The introduction of additional parameters at the partition points reduces the original problem to special Cauchy problems on the subintervals and a block operator equation for the parameters. Under seminorm-boundedness of the coefficient operators, the uniform contraction condition, topological invertibility of the block operator, and contractivity of the coupled parameter iteration, the boundary value problem has a unique solution. The proposed iterative scheme converges to this solution with respect to every seminorm generating the topology of the solution space.

The results show that the Dzhumabaev parameterization method extends naturally from the Banach-space framework to a substantially broader locally convex lattice-algebra setting, provided that the topology and order structure are compatible with the required estimates. The Banach lattice case appears as a direct specialization. The framework also identifies clear directions for further work, including weaker seminorm-dependent contraction principles, verifiable invertibility criteria for the parameter operator, and quantitative error analysis for computational implementations.

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