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Nonlinear Stochastic Impulsive Integro-Differential Equations in Locally Convex Space with Fréchet Topology

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Abstract. *This paper studies an initial value problem for a nonlinear system of first-order stochastic impulsive integro-differential equations whose states take values in a Fréchet space. The model combines Wiener perturbations, a nonlocal integral interaction, and instantaneous state jumps at prescribed impulse times. In contrast to the classical Banach-space setting, the topology of the state space is described by a countable separating family of seminorms, and therefore the solvability analysis must be formulated simultaneously with respect to all seminorms. We introduce a Fréchet space of adapted piecewise continuous stochastic processes endowed with a complete translation-invariant metric generated by a countable family of seminorms. The original impulsive stochastic differential problem is transformed into an equivalent stochastic functional-integral equation. Under seminorm-wise Lipschitz conditions on the drift, diffusion, integral kernel, and impulse operators, together with suitable square-integrability and compactness assumptions, we prove existence of a solution by the Schauder–Tychonoff fixed-point principle. Under a uniform contraction condition with respect to the defining seminorms, uniqueness is established and the successive approximation method is shown to converge in the Fréchet topology. The results provide a framework for extending the theory of nonlinear stochastic impulsive integro-differential equations from Banach spaces to a broad class of complete locally convex spaces.*

Key words: *Fréchet space, stochastic differential equation, impulsive system, integro-differential equation, initial value problem, fixed point, successive approximations, existence and uniqueness.*

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Introduction

Differential equations in infinite-dimensional spaces provide a natural framework for mathematical models in which the state of a system cannot be represented by a finite-dimensional vector. Such situations occur in partial differential equations, hereditary systems, distributed parameter models, evolution equations, and problems involving infinitely many degrees of freedom. Banach spaces are the standard setting for much of the modern theory. Nevertheless, many important spaces of smooth functions, distributions, sequences, and generalized functions are not naturally normable and are more appropriately described as locally convex spaces. Among them, Fréchet spaces occupy a particularly important position because they are complete, metrizable, and locally convex.

The investigation of nonlinear differential equations in Fréchet spaces is substantially more delicate than in Banach spaces. A norm is replaced by a countable family of seminorms, and estimates have to be established simultaneously at every level of the defining topology. In particular, a direct application of the Banach contraction principle is not available unless the Fréchet topology is equipped with a suitable complete metric and the contraction property is verified for that metric. This observation is important when studying nonlinear stochastic equations, where the state variable is random and the solution space must also incorporate measurability, adaptedness, square integrability, and possible discontinuities.

Impulsive differential equations have many applications in various fields of science and technology (see, [1]–[7]). Impulsive differential equations describe systems undergoing instantaneous changes at prescribed moments. Therefore, scientific publications on impulsive differential equations are common [8]–[17]. Modern processes occurring in physics, engineering, biology, economics, ecology, and control theory are often subject to random factors and abrupt short-term changes in state (see, [18]–[30] and [31]). Stochastic perturbations account for random fluctuations, while integral terms describe memory and nonlocal interactions. The combination of these three effects leads to a broad class of stochastic impulsive integro-differential equations. A typical model is

$$dx(t) = f\left(t, x(t), \int_0^T \Theta(t, s, x(s))\, ds\right) dt + g(t, x(t))\, dw(t), \qquad t \in (0, T],$$

with impulse conditions

$$x(t_i^+) - x(t_i^-) = I_i(x(t_i^-)), \qquad i = 1, \ldots, p,$$

and initial condition

$$x(0) = x_0.$$

Here w is a real Wiener process, $0 < t_1 < \cdots < t_p < T$, and the unknown process takes values in a Fréchet space E.

The purpose of this paper is to transfer the fixed-point approach for stochastic impulsive integro-differential equations to the Fréchet-space setting while correcting a fundamental difficulty that arises in a direct Banach-space argument. In particular, an estimate only for two consecutive successive approximations does not imply that the associated operator is a contraction. We therefore formulate the contraction hypothesis for arbitrary pairs of elements of the solution space. In addition, the Fréchet topology is explicitly incorporated into the definition of the stochastic process space through a countable family of seminorms.

The main contribution is twofold. First, an existence theorem is established by applying the Schauder–Tychonoff fixed-point theorem to a continuous compact operator on a closed convex bounded subset of a Fréchet space. Second, under a uniform seminorm-wise contraction condition, uniqueness and convergence of successive approximations are proved by means of a complete metric compatible with the Fréchet topology. This separates the existence mechanism from the uniqueness mechanism and avoids an unjustified use of the Banach contraction principle.

The paper is organized as follows. Section 2 introduces the necessary notions concerning Fréchet spaces, stochastic processes, and the corresponding solution space. Section 3 formulates the stochastic impulsive integro-differential problem and derives its equivalent functional-integral form. Section 4 contains the main existence and uniqueness results. Section 5 discusses successive approximations and a representative example.

1 Preliminary materials

Let E be a real Fréchet space. We assume that its topology is generated by a countable increasing family of continuous seminorms

$$p_1(x) \le p_2(x) \le \cdots, \qquad x \in E,$$

which separates points. Thus,

$$p_n(x) = 0 \quad \text{for all } n \ge 1 \quad \Longrightarrow \quad x = 0.$$

A complete translation-invariant metric compatible with the topology of E is given by

$$d_E(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}. \tag{1.1}$$

Definition 1.1. A topological vector space E is called a Fréchet space if it is locally convex, metrizable, and complete. Equivalently, its topology can be generated by a countable separating family of seminorms and the metric (1.1) is complete.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, and let $w(t)$ be a real-valued Wiener process adapted to $\{\mathscr{F}_t\}$. For an E-valued random variable ξ, define

$$q_n(\xi) = \left(\mathbb{E}\, p_n(\xi)^2\right)^{1/2}, \qquad n \ge 1,$$

whenever the right-hand side is finite.

We consider the space $\mathscr{X}$ of all adapted E-valued processes $x : [0,T] \times \Omega \to E$ which are mean-square continuous on every interval (t_i, t_{i+1}) and possess left limits and right limits at the impulse points. For $x \in \mathscr{X}$, put

$$P_n(x) = \sup_{0\le t\le T} \left(\mathbb{E}\, p_n(x(t))^2\right)^{1/2}.$$

We write

$$\mathscr{X}_2^{PC}(E) = \{x \in \mathscr{X} : P_n(x) < \infty \text{ for every } n \ge 1\}.$$

The topology of this space is generated by the seminorms P_n. We use the metric

$$d_{\mathscr{X}}(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{P_n(x-y)}{1+P_n(x-y)}. \tag{1.2}$$

Under the standard completeness assumptions on the space of mean-square continuous E-valued processes, $\mathscr{X}_2^{PC}(E)$ is a Fréchet space.

Lemma 1.1. *Suppose $x_k \to x$ in $\mathscr{X}_2^{PC}(E)$. Then*

$$P_n(x_k - x) \to 0$$

for every $n \geq 1$. Conversely, if $P_n(x_k - x) \to 0$ for every n, then $x_k \to x$ with respect to the metric (1.2).

For an E-valued predictable process Φ, the stochastic integral $\int_0^t \Phi(s)\, dw(s)$ is understood in the mean-square sense whenever, for every n,

$$\int\limits_0^T \mathbb{E}\, p_n(\Phi(s))^2\, ds < \infty.$$

We assume throughout that the stochastic integral satisfies the seminorm-wise Itô estimate

$$\mathbb{E}\, p_n \left(\int\limits_0^t \Phi(s)\, dw(s) \right)^2 \leq C_n \int\limits_0^t \mathbb{E}\, p_n(\Phi(s))^2\, ds, \tag{1.3}$$

where $C_n > 0$ depends only on the seminorm p_n. In Hilbert-space-valued stochastic integration one may take $C_n = 1$ for the norm induced by the Hilbert structure; in the present abstract setting (1.3) is taken as part of the admissibility assumptions on the stochastic integral.

Lemma 1.2 (Seminorm-wise Gronwall inequality)**.** *Let $r_n : [0,T] \to [0, \infty)$ be locally integrable and suppose*

$$r_n(t) \leq a_n + b_n \int_0^t r_n(s)\, ds,$$

where $a_n, b_n \geq 0$. Then

$$r_n(t) \leq a_n e^{b_n t}, \qquad 0 \leq t \leq T.$$

In particular, if $a_n = 0$, then $r_n(t) = 0$ for all t.

2 Formulation of the problem

Let E be a Fréchet space with defining seminorms $\{p_n\}_{n \geq 1}$. We consider the initial value problem

$$dx(t) = f\left(t, x(t), \int\limits_0^T \Theta(t, s, x(s))\, ds \right) dt + g(t, x(t))\, dw(t), \tag{2.1}$$

for $t \in (0, T] \setminus \{t_1, \ldots, t_p\}$, together with

$$x(t_i^+) - x(t_i^-) = I_i(x(t_i^-)), \qquad i = 1, \ldots, p, \tag{2.2}$$

and

$$x(0) = x_0. \tag{2.3}$$

Here

$$0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T.$$

The mappings are assumed to have the form

$$f : [0, T] \times E \times E \to E, \quad g : [0, T] \times E \to E, \quad \Theta : [0, T]^2 \times E \to E, \quad I_i : E \to E.$$

The integral in (2.1) is a Bochner integral in E whenever it exists. Since E is locally convex, existence of the integral is understood seminorm-wise: for each n, the scalar function $s \mapsto p_n\big(\Theta(t, s, x(s))\big)$ is integrable and the resulting vector-valued integral belongs to E.

Integrating (2.1) on each interval between two consecutive impulse times and incorporating the jumps yields the equivalent functional-integral equation [32]

$$x(t) = x_0 + \int_0^t f\left(s, x(s), \int_0^T \Theta\big(s, \theta, x(\theta)\big)\, d\theta\right) ds + \int_0^t g(s, x(s))\, dw(s) + \sum_{0<t_i<t} I_i(x(t_i^-)). \quad (2.4)$$

Define the operator $\mathscr{A} : \mathscr{X}_2^{PC}(E) \to \mathscr{X}_2^{PC}(E)$ by

$$(\mathscr{A}x)(t) = x_0 + \int_0^t f\left(s, x(s), \int_0^T \Theta(s, \theta, x(\theta))\, d\theta\right) ds + \int_0^t g(s, x(s))\, dw(s) + \sum_{0<t_i<t} I_i(x(t_i^-)). \quad (2.5)$$

Then solving the initial value problem is equivalent to finding a fixed point of $\mathscr{A}$: $\quad \mathscr{A}x = x$.

3 Existence and uniqueness in the Fréchet setting

We impose the following assumptions. For every $n \geq 1$, there exist nonnegative measurable functions $K_{1,n}, K_{2,n}$ on $[0, T]$, a nonnegative measurable function $K_{3,n}$ on $[0, T]^2$, and constants $m_{i,n} \geq 0$ such that, for all $t, s \in [0, T]$ and $x, y, z \in E$,

$$p_n(f(t, x, y) - f(t, z, y)) \leq K_{1,n}(t) p_n(x - z), \quad (3.1)$$

$$p_n(f(t, x, y) - f(t, x, z)) \leq K_{1,n}(t) p_n(y - z), \quad (3.2)$$

$$p_n(g(t, x) - g(t, y)) \leq K_{2,n}(t) p_n(x - y), \quad (3.3)$$

$$p_n(\Theta(t, s, x) - \Theta(t, s, y)) \leq K_{3,n}(t, s) p_n(x - y), \quad (3.4)$$

$$p_n(I_i(x) - I_i(y)) \leq m_{i,n} p_n(x - y). \quad (3.5)$$

Assume also that, for every n,

$$\int_0^T \big[K_{1,n}(s)\big]^2 ds < \infty, \quad \int_0^T \big[K_{2,n}(s)\big]^2 ds < \infty, \quad \sup_{t \in [0,T]} \int_0^T \big[K_{3,n}(t, s)\big]^2 ds < \infty. \quad (3.6)$$

For uniqueness, define

$$\rho_n = 3\left[T \int_0^T \big[K_{1,n}(s)\big]^2 ds + T^2 \int_0^T \int_0^T \big[K_{1,n}(s)\big]^2 \big[K_{3,n}(s, \theta)\big]^2 d\theta ds + \right.$$

$$\left. + C_n \int_0^T \big[K_{2,n}(s)\big]^2 ds + p \sum_{i=1}^p m_{i,n}^2 \right]. \quad (3.7)$$

The precise numerical constant in (3.7) is not essential; it is chosen from the elementary inequality $|a_1+\cdots+a_4|^2 \le 4\sum_{j=1}^4 |a_j|^2$, with constants absorbed into the definition of ρ_n. More generally, any seminorm-wise estimate yielding a constant $\rho_n<1$ is sufficient.

Theorem 3.1 (Existence and uniqueness)**.** *Assume that:*

1. *E is a Fréchet space with defining seminorms $\{p_n\}_{n\ge1}$ and $\mathscr{X}_2^{PC}(E)$ is equipped with the metric* (1.2)*;*
2. *the mappings f, g, Θ, I_i satisfy* (3.1)–(3.5) *and* (3.6)*;*
3. *x_0 is an E-valued, $\mathscr{F}_0$-measurable random variable satisfying $P_n(x_0)<\infty$ for every n;*
4. *for every $n\ge1$, $\rho_n<1$, and in addition there exists $0<\rho<1$ such that $\rho_n\le\rho$ for all n.*

Then the initial value problem (2.1)–(2.3) *has a unique solution $x\in\mathscr{X}_2^{PC}(E)$. Moreover, for every initial approximation $x_0^*\in\mathscr{X}_2^{PC}(E)$, the successive approximations*

$$x_{k+1}=\mathscr{A}x_k, \qquad k=0,1,2,\dots,$$

converge to the unique solution in the Fréchet topology.

Proof. Let $x\in\mathscr{X}_2^{PC}(E)$. The operator $\mathscr{A}$ maps $\mathscr{X}_2^{PC}(E)$ into itself. Indeed, we obtain, for each n

$$\mathbb{E}p_n\big(\mathscr{A}x(t)\big)^2 \le 4\mathbb{E}p_n\big(x(0)\big)^2+4\mathbb{E}\left[\int_0^t p_n\left(f\left(t,x(s),\int_0^T \Theta\left(s,\theta,x(\theta)\right)d\theta\right)\right)ds\right]^2+$$

$$+4\mathbb{E}\left[\int_0^t p_n\big(g(s,x(s))\big)dw(s)\right]^2+4\mathbb{E}\left[\sum_{0<t_i<t} I_i p_n\big(\left(x\left(t_i\right)\right)\big)\right]^2\le$$

$$\le 4\mathbb{E}p_n\big(x_0\big)^2+4tN_f+4tN_g+4pm_I<\infty,$$

where

$$0<N_f=P_n\big(f(t,x,y)\big)^2<\infty;\ 0<N_g=P_n\big(g(t,x)\big)^2<\infty;\ m_I=P_n\big(I(x)\big)^2<\infty.$$

Let $x,y\in\mathscr{X}_2^{PC}(E)$. By (2.5), the difference $\mathscr{A}x-\mathscr{A}y$ is the sum of four terms: the drift integral, the stochastic integral, the integral-kernel contribution contained in the drift, and the impulse sum. Using the elementary inequality for the square of a sum, the Cauchy–Schwarz inequality, (1.3), and assumptions (3.1)–(3.5), we obtain, for each n

$$\mathbb{E}\left[\mathscr{A}x(t)-\mathscr{A}y(t)\right]^2=\mathbb{E}\left[\int_0^t p_n\left(f\left(t,x(s),\int_0^T\Theta\left(s,\theta,x(\theta)\right)d\theta\right)-\right.\right.$$

$$\left.\left.-f\left(t,y(s),\int_0^T\Theta\left(s,\theta,y(\theta)\right)d\theta\right)\right)ds+\right.$$

$$+\int_0^t p_n\left(g(s,x(s))-g(s,y(s))\right)dw(s)+\sum_{0<t_i<t} p_n\left(I_i\left(x\left(t_i\right)\right)-I_i\left(y\left(t_i\right)\right)\right)\Bigg]^2\le$$

$$\le 3\mathbb{E}\left[\int_0^t p_n\left(f\left(t,x(s),\int_0^T\Theta\left(s,\theta,x(\theta)\right)d\theta\right)-f\left(t,y(s),\int_0^T\Theta\left(s,\theta,y(\theta)\right)d\theta\right)\right)ds\right]^2+$$

$$+3\mathbb{E}\left[\int_0^t p_n\left(g(s,x(s))-g(s,y(s))\right)dw(s)\right]^2+$$

$$+3\mathbb{E}\left[\sum_{i=1}^p p_n\left(I_i\left(x\left(t_i\right)\right)-I_i\left(y\left(t_i\right)\right)\right)\right]^2\le$$

$$\le 3\mathbb{E}t\int_0^t p_n\left(f\left(t,x(s),\int_0^T\Theta\left(s,\theta,x(\theta)\right)\,d\theta\right)-f\left(t,y(s),\int_0^T\Theta\left(s,\theta,y(\theta)\right)d\theta\right)\right)^2ds+$$

$$+3\mathbb{E}\int_0^t p_n\left(g(s,x(s))-g(s,y(s))\right)^2ds+$$

$$+3p\mathbb{E}\sum_{i=1}^p p_n\left(I_i\left(x\left(t_i\right)\right)-I_i\left(y\left(t_i\right)\right)\right)^2\le$$

$$\le 3\sup_{0\le t\le T}\left|t\int_0^t\left[p_n\left(K_1(s)\right)^2+\int_0^T p_n\left(K_3(s,\theta)\right)^2d\theta\right]ds\right|\mathbb{E}p_n\left(x(t)-y(t)\right)^2+$$

$$+3\sup_{0\le t\le T}\left|\int_0^t p_n\big(K_2(s)\big)^2ds\right|\mathbb{E}p_n\left(x(t)-y(t)\right)^2+$$

$$+3p\sum_{i=1}^p m_i\mathbb{E}p_n\left(x(t)-y(t)\right)^2\le\rho\cdot P_n\left(x(t)-y(t)\right)^2,$$

where

$$\rho=3\sup_{0\le t\le T}\left\{\left|t\int_0^t\big[\left(K_1(s)\right)^2+\int_0^T\left(K_3(s,\theta)\right)^2d\theta\big]ds\right|+\left|\int_0^t\big(K_2(s)\big)^2ds\right|+p\sum_{i=1}^p\left(m_i\right)^2\right\}.$$

So, we obtain that

$$P_n(\mathscr{A}x-\mathscr{A}y)^2\le\rho_n P_n(x-y)^2.$$

Consequently,

$$P_n(\mathscr{A}x-\mathscr{A}y)\le\sqrt{\rho}\,P_n(x-y),\qquad n\ge 1.$$

Put $q=\sqrt{\rho}<1$. Since the function $r\mapsto r/(1+r)$ is increasing and

$$\frac{qr}{1+qr}\le q\frac{r}{1+r},\qquad r\ge 0,$$

we have

$$d_{\mathscr{X}}(\mathscr{A}x, \mathscr{A}y) \le q\, d_{\mathscr{X}}(x, y).$$

Thus $\mathscr{A}$ is a contraction on the complete metric space $\mathscr{X}_2^{PC}(E)$. The Banach fixed-point theorem yields a unique fixed point $x \in \mathscr{X}_2^{PC}(E)$, which is the unique solution of (2.4), and hence of the initial value problem (2.1)–(2.3). Moreover,

$$d_{\mathscr{X}}(x_k, x) \le q^k d_{\mathscr{X}}(x_0^*, x),$$

so the successive approximations converge to x in the Fréchet topology. □

Remark 3.1. The proof above differs essentially from an argument based only on consecutive iterates. To establish that $\mathscr{A}$ is a contraction, one must prove an estimate for arbitrary $x, y \in \mathscr{X}_2^{PC}(E)$:

$$P_n(\mathscr{A}x - \mathscr{A}y) \le qP_n(x - y), \qquad n \ge 1,$$

with one common $q < 1$.

Remark 3.2. When $\Theta(t, s, x(s))$ is zero for all t and s, we can be use the Seminorm-wise Gronwall inequality in the proof of uniqueness of the solution to initial value problem (2.1)–(2.3).

Theorem 3.2 (Existence under compactness)**.** *Assume that conditions 1–3 of the preceding theorem hold. The operator $\mathscr{A}$ maps $\mathscr{X}_2^{PC}(E)$ into itself. Replace the uniform contraction condition by the following assumptions: there exists a closed, convex, bounded subset $\mathscr{K} \subset \mathscr{X}_2^{PC}(E)$ such that $\mathscr{A}(\mathscr{K}) \subset \mathscr{K}$, the operator $\mathscr{A} : \mathscr{K} \to \mathscr{K}$ is continuous, and $\mathscr{A}(\mathscr{K})$ is relatively compact in the Fréchet topology.*

Then the initial value problem (2.1)–(2.3) *has at least one solution in $\mathscr{K}$.*

Proof. The set $\mathscr{K}$ is a nonempty closed convex subset of the Fréchet space $\mathscr{X}_2^{PC}(E)$. By assumption, $\mathscr{A}$ is continuous and maps $\mathscr{K}$ into itself, while $\mathscr{A}(\mathscr{K})$ is relatively compact. The Schauder–Tychonoff fixed-point theorem therefore guarantees the existence of $x \in \mathscr{K}$ such that $\mathscr{A}x = x$. By the equivalence between (2.1)–(2.3) and (2.4), this fixed point is a solution of the initial value problem. □

Remark 3.3. The second theorem is useful when the operator is not contractive. In a general Fréchet space, boundedness alone does not imply relative compactness. Therefore, a compactness hypothesis, or a concrete criterion implying relative compactness, must be verified separately. Typical sufficient conditions may involve seminorm-wise equicontinuity together with compactness of the ranges of the coefficient operators in the relevant topology.

4 Successive approximations and a model example

Under the hypotheses of the first theorem, choose $x_0^* \in \mathscr{X}_2^{PC}(E)$ and define $x_{k+1} = \mathscr{A}x_k$. Then

$$d_{\mathscr{X}}(x_{k+1}, x_k) \le q^k d_{\mathscr{X}}(x_1, x_0^*),$$

where $q < 1$. Hence the sequence is Cauchy in the complete metric (1.2). Since the metric is compatible with the Fréchet topology, the sequence converges to the unique solution.

The following example illustrates the role of the seminorm structure. Let $E = C^\infty([0,1])$ with seminorms

$$p_n(u) = \max_{0\le j\le n}\max_{0\le r\le 1}|u^{(j)}(r)|.$$

Then E is a Fréchet space. Consider

$$dx(t,r) = \left[a(t,r)x(t,r) + b(t,r)\int_0^1 c(r,s)x(t,s)\,ds + h(t,r)\right]dt + \sigma(t,r)x(t,r)\,dw(t),$$

with impulse conditions

$$x(t_i^+,r) - x(t_i^-,r) = \lambda_i(r)x(t_i^-,r)$$

and initial condition $x(0,r) = x_0(r)$. Suppose that all coefficients are smooth in r and satisfy, for each n, estimates of the form

$$p_n(a(t,\cdot)u) \le A_n(t)p_n(u), \qquad p_n(\sigma(t,\cdot)u) \le S_n(t)p_n(u),$$

$$p_n\left(\int_0^1 c(\cdot,s)u(s)ds\right) \le C_n p_n(u), \qquad p_n(\lambda_i u) \le M_{i,n}p_n(u).$$

If the corresponding constants satisfy the uniform contraction condition of Theorem 4.1, then the problem possesses a unique solution in $\mathscr{X}_2^{PC}(C^\infty([0,1]))$, and the successive approximations converge simultaneously with respect to every seminorm P_n.

This example shows why the Fréchet framework is natural for systems in which one needs to control arbitrarily high spatial derivatives. A single Banach norm may fail to capture the full regularity structure, whereas the countable family $\{p_n\}$ records all derivative levels simultaneously.

Discussion

The passage from Banach spaces to Fréchet spaces changes the functional-analytic structure of the solvability problem in an essential way. The principal difficulty is not the formal appearance of the stochastic or impulsive terms but the absence of a single norm controlling the entire topology. The solution must therefore be estimated at every seminorm level.

The first theorem provides a clean route to existence and uniqueness when the Lipschitz constants are uniformly contractive across all seminorms. The uniformity condition is important. If one only knows $\rho_n < 1$ separately for each n, with $\rho_n \to 1$, the resulting operator need not be a contraction in the metric (1.2). In that situation, the Schauder–Tychonoff approach or another fixed-point method is more appropriate.

The second theorem separates existence from uniqueness. Compactness can yield existence even when no contraction is available, while a separate seminorm-wise Lipschitz argument may then provide uniqueness. This distinction is particularly relevant for nonlinear integral operators, for which compactness may be easier to establish than a global contraction estimate.

The Fréchet formulation also clarifies the role of regularity. For $E = C^\infty([0,1])$, for example, the seminorms control derivatives of increasing order. The stochastic, integral, and impulse operators must therefore preserve the entire scale of regularity. This requirement can lead to conditions that are stronger than those needed in a fixed Banach space but are natural when the desired solution is smooth.

The framework can be extended in several directions. One may replace C^∞ by spaces of analytic functions, Schwartz spaces, spaces of test functions, or projective limits of Banach spaces. More general locally convex spaces can also be considered, although additional assumptions are then required because metrizability and completeness may fail. Another direction is to investigate stochastic equations driven by cylindrical Wiener processes or more general martingale noise. Finally, periodic boundary conditions, delay terms, fractional operators, and optimal control problems can be incorporated into the same functional-analytic framework.

Conclusion

We have developed a Fréchet-space framework for the initial value problem associated with a nonlinear stochastic impulsive integro-differential equation. The model incorporates three nonlocal mechanisms: stochastic perturbations, integral memory, and instantaneous impulses. The original differential problem is transformed into an equivalent stochastic functional-integral equation acting on a Fréchet space of adapted piecewise continuous processes.

The main results distinguish two complementary solvability mechanisms. Under a uniform contraction condition formulated simultaneously with respect to the defining seminorms, the associated operator is a contraction in a complete metric compatible with the Fréchet topology. This yields existence, uniqueness, and convergence of successive approximations. Without a contraction condition, existence can instead be obtained from the Schauder–Tychonoff fixed-point theorem under continuity, invariance, and relative compactness assumptions.

A central methodological point is that estimates for consecutive successive approximations are insufficient to prove contractivity of an operator. The correct contraction estimate must hold for arbitrary pairs of processes. By formulating the problem in this way, the present approach avoids the logical gap that may arise in a direct adaptation of Banach-space arguments.

The obtained framework provides a basis for further research on nonlinear stochastic equations in spaces of smooth functions and other Fréchet spaces. Future work may address local and global solvability under weaker seminorm-wise assumptions, stability with respect to initial data, periodic solutions, stochastic control problems, and equations with delay or fractional operators.

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References

1. M. Benchohra, J. Henderson, and S. K. Ntouyas, *Impulsive differential equations and inclusions. Contemporary mathematics and its application* Hindawi Publishing Corporation, New York, 2006.

2. A. Halanay, and D. Veksler, *Qualitative theory of impulsive systems.* Mir, Moscow, 1971. 309 p. (in Russian).

3. V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of impulsive differential equations.* World Scientific, Singapore, 1989.

4. N. A. Perestyk, V. A. Plotnikov, A. M. Samoilenko, N. V. Skripnik, *Differential equations with impulsive effect: Multivalued Right-Hand Sides with Discontinuities.* DeGruyter Stud. 40, Math. Walter de Gruter Co., Berlin, 2011.

5. A. M. Samoilenko and N. A. Perestyk, *Impulsive differential equations.* World Sci., Singapore, 1995.

6. I. Stamova and G. Stamov, *Impulsive biological models.* In: Applied impulsive mathematical models. CMS Books in Mathematics. Springer, Cham., 2016.

7. J. Catlla, D. G. Schaeffer, Th. P. Witelski, E. E. Monson, A. L. Lin, "On spiking models for synaptic activity and impulsive differential equations", *SIAM Review* **50** (3), 553–569 (2008).

8. A. N. Abdullozhonova, T. K. Yuldashev, A. K. Fayziyev, "Mixed problem for an impulsive parabolic integro-dfferential equation with involution and nonlinear conditions", *Lobachevskii J. Math.* **45** (3), 899–911 (2024). https://doi.org/10.1134/S199508022460078X

9. A. Anguraj, M. M. Arjunan, "Existence and uniqueness of mild and classical solutions of impulsive evolution equations", *Elect. J. Differential Equations* **2005** (111), 1–8 (2005).

10. A. Ashyralyev, Ya. A. Sharifov, "Existence and uniqueness of solutions for nonlinear impulsive differential equations with two-point and integral boundary conditions", *Advances in Difference Equations* **2013** (173) (2013).

11. Ch. Bai, D. Yang, "Existence of solutions for second-order nonlinear impulsive differential equations with periodic boundary value conditions", *Boundary Value Problems* Hindawi Publishing Corporation, **2007** (41589), 1–13 (2007).

12. L. Bin, L. Xinzhi, and L. Xiaoxin, "Robust global exponential stability of uncertain impulsive systems", *Acta Mathematika Scientia* **25** (1), 161–169 (2005).

13. M. J. Mardanov, Ya. A. Sharifov, M. H. Habib, "Existence and uniqueness of solutions for first-order nonlinear differential equations with two-point and integral boundary conditions", *Electr. J. Differential Equations* **2014** (259), 1–8 (2014).

14. T. K. Yuldashev, A. K. Fayziyev, "Inverse problem for a second order impulsive system of integro-differential equations with two redefinition vectors and mixed maxima", *Nanosystems: Phys. Chem. Math.* **14** (1), 13–21 (2023). https://doi.org/10.17586/2220-8054-2023-14-1-13-21

15. T. K. Yuldashev, T. A. Abduvahobov, "Periodic solutions for an impulsive system of fractional order integro-differential equations with maxima", *Lobachevskii J. Math.* **44** (10), 4401–4409 (2023). https://doi.org/10.1134/S1995080223100451

16. T. K. Yuldashev, A. K. Fayziyev, "Determination of the coefficient function in a Whitham type nonlinear differential equation with impulse effects", *Nanosystems: Phys. Chem. Math.* **14** (3), 312–320 (2023). https://doi.org/10.17586/2220-8054-2023-14-3-312-320

17. T. K. Yuldashev, A. K. Fayziyev, "Mixed problem for a linear differential equation of parabolic type with nonlinear impulsive conditions", *Nanosystems: Phys. Chem. Math.* **15** (2), 160–169 (2024). https://doi.org/10.17586/2220-8054-2024-15-2-160-169

18. B. Oksendal, *Stochastic differential equations: An introduction with applications.* 6th Edition, Springer, Berlin, 2003.

19. R. Z. Khasminskii, *Stochastic stability of differential equations.* 2nd Edition, Springer, Heidelberg, 2012.

20. N. Ikeda, S. Watanabe, S. *Stochastic differential equations and diffusion processes.* 2nd Edition, North-Holland Mathematical Library, Elsevier, Amsterdam, 1989.

21. X. Mao, *Stochastic differential equations and applications.* 2nd Edition, Horwood Publishing, Chichester, 2007.

22. L. C. G. Rogers, D. Williams, *Diffusions, Markov processes and martingales.* Vol. 2: Itô Calculus. 2nd Edition, Cambridge University Press, 2000.

23. K. Itô, *On a stochastic integral equation. Proceedings of the Japan Academy* **22**, 32–35 (1946).

24. K. Itô, "On a Formula Concerning Stochastic Differentials", *Nagoya Mathematical Journal* **3**, 55–65 (1951).

25. K. Itô, H. P. McKean, *Diffusion processes and their sample paths.* Springer-Verlag, Berlin, 1965.

26. K. Itô, *Foundations of stochastic differential equations in infinite dimensional spaces.* SIAM, Philadelphia, 1984.

27. J. Jacod, "Weak and strong solutions of stochastic differential equations", *Stochastics* **3**, 171–191 (1980).

28. H. Kunita, *Stochastic flows and stochastic differential equations.* Cambridge University Press, 1990.

29. A. Y. Veretennikov, "On strong solutions and explicit formulas for solutions of stochastic integral equations", *Mathematics of the USSR Sbornik* **39** (3), 387–403 (1981). [in Russian]

30. T. Yamada, S. Watanabe, "On the uniqueness of solutions of stochastic differential equations", *Journal of Mathematics of Kyoto University* **11** (1), 155–167 (1971).

31. I. I. Gilhman, A. V. Skorokhod, *Stochastic differential equations.* Naukovo dumka, Kiev, 1968. [in Russian]

32. T. K. Yuldashev, K. Z. Negmatova, "Initial value problem for the nonlinear stochastic impulsive integro-differential equations", *Journal of Osh University "Differential Equations"* **1** (2), 70–79 (2026). https://doi.org/10.52754/16949242 2026 2 6