ISSN: 1694-9242

Journal of Osh University "Differential Equations", 2026, Volume 1, Issue 2, P. 97–115.

https://doi.org/10.52754/16949242_2026_2_8

UDC 517.953.5

Boundary Value Problem for the Fifth Order Partial Differential Equations

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Received: May 29, 2026; **First on-line published**: August 15, 2026

Abstract. *We investigate a boundary value problem for a fifth-order partial differential equation with a real parameter in a rectangular domain. The equation contains third-order differentiation with respect to time and a mixed fifth-order derivative with respect to the temporal and spatial variables. The problem is supplemented by homogeneous boundary conditions of Samarskii–Ionkin type with respect to the spatial variable and by three nonstandard two-point conditions with respect to time. The principal objective is to establish a constructive representation of the solution, determine the parameter values for which the problem is regular, and justify the convergence of the resulting Fourier series.*

The method of separation of variables reduces the spatial part of the problem to a Sturm–Liouville spectral problem. Its eigenvalues and normalized eigenfunctions are obtained explicitly, and the eigenfunctions form a complete orthonormal system in $L_2(0,1)$. Expansion of the unknown solution and the right-hand side with respect to this system transforms the original partial differential equation into a countable family of third-order ordinary differential equations. These equations are solved by the method of variation of parameters. The three temporal boundary conditions lead, for each Fourier mode, to a finite-dimensional linear algebraic system whose characteristic determinant depends on the parameter ω.

The zeros of the corresponding characteristic expression determine an exceptional set of parameter values. Outside this set, the algebraic systems are uniquely solvable and an explicit Fourier representation of the solution is obtained. It is proved that, for regular values of the parameter, the boundary value problem can have at most one solution. Under additional smoothness and compatibility assumptions on the boundary data and the right-hand side, estimates of the Fourier coefficients together with Bessel's inequality yield absolute and uniform convergence of the solution series and of the derivatives required by the differential equation. Consequently, the formal Fourier representation defines a classical solution of the boundary value problem. The obtained results clarify the role of the spectral parameter in the solvability of higher-order boundary value problems and provide a constructive framework that can be adapted to related partial differential equations with nonstandard temporal and spatial conditions.

Key words: *fifth-order partial differential equation; boundary value problem; spectral parameter; Fourier method; eigenfunctions; regular parameter values; uniqueness; classical solvability; Samarskii–Ionkin type conditions.*

MSC 2020: 35A01, 35A02, 35S15.

Introduction

Higher-order partial differential equations arise in a variety of problems of mathematical physics and continuum mechanics. Such equations appear, in particular, in mathematical models involving filtration processes, dispersive phenomena, elasticity, hydrodynamics, and higher-order evolutionary effects. Their mathematical analysis is often considerably more complicated than that of standard second-order equations because the order of differentiation, the interaction between temporal and spatial derivatives, and the form of the prescribed boundary conditions strongly influence both solvability and regularity of solutions.

An important class of such problems is formed by partial differential equations containing a spectral or physical parameter. In this situation, solvability may depend essentially on the value of the parameter: for certain exceptional values the corresponding algebraic or operator equations may become singular, whereas outside this exceptional set the problem may possess a unique solution. Therefore, one of the central questions is not only to construct a formal solution but also to identify the set of regular parameter values and to establish conditions under which the formal representation actually defines a classical solution.

Boundary conditions of nonstandard type create an additional difficulty. Classical initial or Dirichlet conditions frequently lead to well-established self-adjoint spectral problems and standard Fourier expansions. In contrast, mixed conditions prescribed at different points of the temporal interval may produce characteristic determinants whose zeros determine resonance-type parameter values. This makes it necessary to combine spectral analysis, ordinary differential equation techniques, finite-dimensional algebraic systems, and convergence estimates for Fourier series

$$\left[\frac{\partial^3}{\partial t^3}+\frac{\partial^5}{\partial t^3\partial x^2}+\omega^3\frac{\partial^2}{\partial x^2}\right]U(t,x)=f(t,x),$$

where $T>0$ is fixed and $\omega\neq 0$ is a real parameter.

With respect to the spatial variable, the equation is supplemented by the boundary conditions

$$U(t,1)=0,\qquad U_x(t,0)=0,$$

while the temporal variable is subject to the conditions

$$U(T,x)=\varphi_1(x),\qquad U_t(0,x)=\varphi_2(x),\qquad U(0,x)+U_{tt}(0,x)=\varphi_3(x).$$

The data are assumed to satisfy the natural compatibility conditions inherited from the spatial boundary conditions.

The structure of the differential operator makes the Fourier method particularly effective. Applying separation of variables to the homogeneous spatial part gives the spectral problem

$$\vartheta''(x)+\lambda\vartheta(x)=0,\qquad \vartheta(1)=0,\qquad \vartheta'(0)=0.$$

Its eigenvalues are

$$\lambda_n=\left(\frac{\pi}{2}+\pi n\right)^2,\qquad n=0,1,2,\ldots,$$

and the corresponding normalized eigenfunctions are

$$\vartheta_n(x)=\sqrt{2}\cos(\sqrt{\lambda_n},x).$$

These functions form a complete orthonormal system in $L_2(0,1)$. Consequently, both the unknown function and the right-hand side can be expanded in Fourier series with respect to this system.

The Fourier transformation reduces the original fifth-order partial differential equation to a countable family of third-order ordinary differential equations of the form

$$u_n'''(t) + (2\mu_n(\omega))^3 u_n(t) = g_n(t),$$

where $\mu_n(\omega)$ depends explicitly on ω and on the spatial eigenvalue λ_n. The homogeneous equation possesses one real exponential mode and two oscillatory exponential modes. Using variation of parameters, an explicit representation of $u_n(t)$ is derived. The constants occurring in this representation are then determined from the three temporal conditions.

A central feature of the analysis is the determinant of the resulting 3×3 algebraic system. Its vanishing corresponds to singular or resonant values of the parameter. Analysis of the characteristic equation allows us to separate the exceptional parameter set from the regular set. For parameter values belonging to the regular set, the algebraic system is uniquely solvable for every Fourier mode, and the coefficients of the solution can therefore be written explicitly.

The first principal result of the paper establishes uniqueness of the solution for regular parameter values. Indeed, if two solutions existed, their difference would solve the corresponding homogeneous problem with zero boundary data and zero right-hand side. All Fourier coefficients of this difference would vanish, and completeness of the eigenfunction system would then imply that the difference itself is identically zero.

The construction of a formal Fourier series, however, does not by itself establish classical solvability. It is also necessary to justify convergence of the series and of the derivatives appearing in the original differential equation. For this purpose, additional smoothness assumptions are imposed on the functions φ_i and f. Repeated integration by parts yields appropriate decay estimates for their Fourier coefficients. These estimates, combined with Bessel's inequality and bounds for the temporal coefficient functions and the integral kernel, provide absolute and uniform convergence of the relevant series.

Thus, the purpose of the present work is threefold: to obtain an explicit Fourier representation of the solution, to identify the regular values of the parameter ω, and to establish sufficient smoothness conditions ensuring that the formal solution is a classical one. The resulting analysis develops further the Fourier approach to higher-order partial differential equations with parameters and nonstandard boundary conditions and provides a basis for studying related problems with more general differential operators and boundary data.

The obtained results extend earlier investigations devoted to higher-order equations with classical boundary conditions and demonstrate the effectiveness of spectral methods in the study of nonlinear nonlocal equations with Samarskii–Ionkin type constraints. The developed approach may also be useful for a wider class of nonlocal evolution problems governed by higher-order differential operators.

However, differential equations of mathematical physics have a wide range of applications in the sciences and technology (see, for examples [1]–[8]). According to applications this field of mathematics is developing at a rapid pace and a lot of works are being published (see, for examples [9]–[15]).

The main equations of the theory of non-stationary filtration in fractured-pore formations are formulated in the work of G. I. Barenblatt, Yu. P. Zheltov and I. N. Kochina [16] (see, also

[17]) and, further, developed by many authors [18]–[23]. Boussinesq type partial differential equations with boundary conditions are studied partially, in [24]–[27].

The spectral problems are considered in the works [28]–[31].

In the works [32]–[34], the similar equation is studied by Cauchy and Dirichlet conditions. So, our new work, which presents in this paper, is further development of the works [32]–[34].

1 Formulation of the problem statement

In this paper, we study the solvability of the mixed problem for a fifth order differential equation. In the rectangular domain, $\Omega = \{0 < t < T,\ 0 < x < 1\}$ we consider the following partial differential equation

$$\left[\frac{\partial^3}{\partial t^3} - \frac{\partial^5}{\partial t^3 \partial x^2} - \omega^3 \frac{\partial^2}{\partial x^2}\right] U(t,x) = f(t,x), \tag{1.1}$$

where $f(t,x) \in C(\bar{\Omega})$, $\bar{\Omega} \leq \{0 \leq t \leq T,\ \ 0 \leq x \leq 1\}$, T is given positive number, ω is nonzero real parameter.

In solving partial differential equation (1.1), with respect to spatial variable x we use Samarskii-Ionkin type boundary value conditions

$$U(t,1) = 0\,, \quad U_x(t,0) = 0\,, \quad 0 \leq t \leq T, \tag{1.2}$$

With respect to time variable t we use another boundary value conditions

$$U(T,x) = \varphi_1(x), \quad U_t(0,x) = \varphi_2(x), \quad U(0,x) + U_{tt}(0,x) = \varphi_3(x), \tag{1.3}$$

where $\varphi_i(x)$ are enough smooth functions on the segment $[0,1]$.

For this function the following conditions are fulfilled $\varphi_i(1) = \varphi_i'(0) = 0,\ \ i = 1,2,3.$

Problem statement. To find a function $U(t,x) \in C(\overline{\Omega}) \cap C^{3,2}_{t,x}(\Omega)$, which satisfies differential equation (1.1) and the specified conditions (1.2) and (1.3).

2 Formal solution of the problem (1.1)-(1.3)

We will look for a non-trivial particular solution of the equation in the form $U(t,x) = u(t)\vartheta(x)$. Substituting this product of functions, depending from different variables, into equation (1.1), we obtain

$$\frac{u'''(t)}{u'''(t) + \omega^3 u(t)} = \frac{\vartheta''(x)}{\vartheta(x)}.$$

Hence, we obtain ordinary differential equation

$$\vartheta''(x) + \lambda\vartheta(x) = 0, \quad 0 < \lambda$$

with boundary conditions

$$\vartheta(1) = \vartheta'(0) = 0.$$

Eigenfunctions and eigenvalues of this spectral problem has the forms:

$$\vartheta_n\left(x\right) = \sqrt{2}\cos\sqrt{\lambda_n}x,\ \ \lambda_n = \left(\frac{\pi}{2} + \pi n\right)^2, \quad n = 0,1,2,...,$$

These eigenfunctions are complete and orthonormal in the space $L_2[0,1]$.

The solution of the problem (1.1)–(1.3) we search in the form of Fourier series

$$U(t,x)=\sum_{n=1}^{\infty}u_n(t)\,\vartheta_n(x), \tag{2.1}$$

where

$$\vartheta_n\left(x\right)=\sqrt{2}\cos\sqrt{\lambda_n}x,\quad \lambda_n=\left(\frac{\pi}{2}+\pi n\right)^2,\qquad n=0,1,2,...,$$

$$u_n(t)=\int\limits_0^1 U(t,x)\,\vartheta_n(x)dx. \tag{2.2}$$

Substituting the Fourier series (2.1) into the given differential equation (1.1), we obtain a countable system of third order ordinary differential equations

$$u_n'''(t)+\frac{\lambda_n}{1+\lambda_n}u_n(t)=\frac{1}{1+\lambda_n}f_n(t), \tag{2.3}$$

where

$$f_n(t)=\int\limits_0^1 f(t,x)\vartheta_n(x)dx.$$

For the Fourier coefficients (2.2) the differential equation (2.3) we rewrite as

$$u_n'''(t)+\left(2\mu_n(\omega)\right)^3u_n(t)=g_n(t), \tag{2.4}$$

where

$$\mu_n(\omega)=\frac{\omega}{2}\sqrt[3]{\frac{\lambda_n}{1+\lambda_n}}\text{and}g_n(t)=\frac{1}{1+\lambda_n}f_n(t). \tag{2.5}$$

The characteristic equation $k^3+(2\mu_n(\omega))^3=0$ for the homogeneous equation

$$u_n'''(t)+(2\mu_n(\omega))^3u_n(t)=0 \tag{2.6}$$

has the roots

$$k_1=-2\mu_n,\quad k_{2/3}=\left(1\pm\sqrt{3}i\right)\mu_n.$$

The system of fundamental solutions of the homogeneous equation (2.6) consists of the following functions

$$u_{1,n}(t)=e^{-2\mu_nt},\qquad u_{2,n}(t)=e^{\mu_nt}\cos\sqrt{3}\mu_nt,\qquad u_{3,n}(t)=e^{\mu_nt}\sin\sqrt{3}\mu_nt. \tag{2.7}$$

Lemma 2.1. *For the system of fundamental solutions* (2.7) *the following*

$$\left(u_{1,n}(t)\right)^{(n)}=\left(-2\mu_n\right)^nu_{1,n}(t)\,and\begin{pmatrix}u_{2,n}(t)\\u_{3,n}(t)\end{pmatrix}^{(n)}=\mu_n^nA^n \tag{2.8}$$

equalities hold, where $A=\begin{pmatrix}1&-\sqrt{3}\\\sqrt{3}&1\end{pmatrix}$, *which*

$$A^{4k}=(-8)^kA^k,\quad k\in\mathbb{N}$$

also satisfies the equality.

The *proof* is done by mathematical induction.

According to Lemma 2.1, we calculate up to the third-order derivative of the system of fundamental solutions (2.7):

$$\left[\begin{array}{l} u'_{1,n}(t) = -2\mu_n u_{1,n}(t), \\ u'_{2,n}(t) = \mu_n u_{2,n}(t) - \sqrt{3}\mu_n u_{3,n}(t), \\ u'_{3,n}(t) = \sqrt{3}\mu_n u_{2,n}(t) + \mu_n u_{3,n}(t); \end{array}\right. \quad \left[\begin{array}{l} u''_{1,n}(t) = 4\mu_n^2 u_{1,n}(t), \\ u''_{2,n}(t) = -2\mu_n^2 u_{2,n}(t) - 2\sqrt{3}\mu_n^2 u_{3,n}(t), \\ u''_{3,n}(t) = 2\sqrt{3}\mu_n^2 u_{2,n}(t) - 2\mu_n^2 u_{3,n}(t); \end{array}\right.$$

$$\left[\begin{array}{l} u'''_{1,n}(t) = -8\mu_n^3 u_{1,n}(t), \\ u'''_{2,n}(t) = -8\mu_n^3 u_{2,n}(t), \\ u'''_{3,n}(t) = -8\mu_n^3 u_{3,n}(t). \end{array}\right.$$

It is known that the solution of the homogeneous equation (2.6) is of the form

$$\tilde{u}_n(t) = A_{1,n} u_{1,n}(t) + A_{2,n} u_{2,n}(t) + A_{3,n}\, u_{3,n}(t) \tag{2.9}$$

and to find the solution of the non-homogeneous equation (2.4) we obtain by variation method of variables

$$u_n(t) = A_{1,n}(t) u_{1,n}(t) + A_{2,n}(t) u_{2,n}(t) + A_{3,n}(t)\, u_{3,n}(t). \tag{2.10}$$

Unknown functions $A_{i,n}(t),\quad i = 1, 2, 3$ in the equation (2.10)

$$\begin{cases} A'_{1,n}(t) u_{1,n}(t) + A'_{2,n}(t) u_{2,n}(t) + A'_{3,n}(t) u_{3,n}(t) = 0, \\ A'_{1,n}(t) u'_{1,n}(t) + A'_{2,n}(t) u'_{2,n}(t) + A'_{3,n}(t)\, u'_{3,n}(t) = 0, \\ A'_{1,n}(t) u''_{1,n}(t) + A'_{2,n}(t) u''_{2,n}(t) + A'_{3,n}(t)\, u''_{3,n}(t) = g_n(t) \end{cases}$$

we find from the system of algebraic differential equations:

$$W[u_{1,n}, u_{2,n}, u_{3,n}] = \begin{vmatrix} u_{1,n}(t) & u_{2,n}(t) & u_{3,n}(t) \\ u'_{1,n}(t) & u'_{2,n}(t) & u'_{3,n}(t) \\ u''_{1,n}(t) & u''_{2,n}(t) & u''_{3,n}(t) \end{vmatrix} = 12\sqrt{3}\mu_n^3,$$

$$\Delta_{A'_{1,n}} = \begin{vmatrix} 0 & u_{2,n}(t) & u_{3,n}(t) \\ 0 & u'_{2,n}(t) & u'_{3,n}(t) \\ g_n(t) & u''_{2,n}(t) & u''_{3,n}(t) \end{vmatrix} = \frac{\sqrt{3}\mu_n}{u_{1n}(t)} g_n(t) = \sqrt{3}\mu_n e^{2\mu_n t} g_n(t),$$

$$\Delta_{A'_{2,n}} = \begin{vmatrix} u_{1,n}(t) & 0 & u_{3,n}(t) \\ u'_{1,n}(t) & 0 & u'_{3,n}(t) \\ u''_{1,n}(t) & g_n(t) & u''_{3,n}(t) \end{vmatrix} = \left(-3\mu_n u_{1,n}(t) u_{3,n}(t) - \sqrt{3}\mu_n u_{1,n}(t) u_{2,n}(t)\right) g_n(t) =$$

$$= -\sqrt{3}\mu_n e^{-\mu_n t}\left(\sqrt{3}\sin\sqrt{3}\mu_n t + \cos\sqrt{3}\mu_n t\right) g_n(t) = -2\sqrt{3}\mu_n e^{-\mu_n t}\sin\left(\sqrt{3}\mu_n t + \frac{\pi}{6}\right),$$

$$\Delta_{A'_{3,n}} = \begin{vmatrix} u_{1,n}(t) & u_{2,n}(t) & 0 \\ u'_{1,n}(t) & u'_{2,n}(t) & 0 \\ u''_{1,n}(t) & u''_{2,n}(t) & g_n(t) \end{vmatrix} = \left(3\mu_n u_{1,n}(t) u_{2,n}(t) - \sqrt{3}\mu_n u_{1,n}(t) u_{3,n}(t)\right) g_n(t) =$$

$$= \sqrt{3}\mu_n e^{-\mu_n t}\left(\sqrt{3}\cos\sqrt{3}\mu_n t - \sin\sqrt{3}\mu_n t\right) g_n(t) = 2\sqrt{3}\mu_n e^{-\mu_n t}\cos\left(\sqrt{3}\mu_n t + \frac{\pi}{6}\right).$$

According to Cramer's formula, we find $A'_{i,n}(t),\quad i = 1, 2, 3$:

$$A'_{1,n}(t) = \frac{e^{2\mu_n t}}{12\mu_n^2} g_n(t), A'_{2,n}(t) = -\frac{e^{-\mu_n t}\sin\left(\sqrt{3}\mu_n t + \frac{\pi}{6}\right)}{6\mu_n^2} g_n(t),$$

$$A'_{3,n}(t) = \frac{e^{-\mu_n t}\cos\left(\sqrt{3}\mu_n t + \frac{\pi}{6}\right)}{6\mu_n^2} g_n(t).$$

We integrate each $A'_{i,n}(t),\ \ i=1,2,3$ on the interval $(0,t)$ to find $A_{i,n}(t),\ \ i=1,2,3$:

$$A_{1,n}(t) = \frac{1}{12\mu_n^2}\int_0^t e^{2\mu_n s} g_n(s)ds + B_{1,n}, \tag{2.11}$$

$$A_{2,n}(t) = -\frac{1}{6\mu_n^2}\int_0^t e^{-\mu_n s}\sin\left(\sqrt{3}\mu_n s + \frac{\pi}{6}\right) g_n(s)ds + B_{2,n}, \tag{2.12}$$

$$A_{3,n}(t) = \frac{1}{6\mu_n^2}\int_0^t e^{-\mu_n s}\cos\left(\sqrt{3}\mu_n s + \frac{\pi}{6}\right) g_n(s)ds + B_{3,n}, \tag{2.13}$$

where $B_{i,n}(t),\ \ i=1,2,3$ are constants (2.11)–(2.13), we take them to (2.10) and simplify to obtain the following general solution:

$$U(t,x) = \sum_{n=1}^{\infty}\left[\frac{1}{12\mu_n^2}\int_0^t \left(e^{2\mu_n(s-t)} - 2e^{\mu_n(t-s)}\sin\left(\sqrt{3}\mu_n\left(s-t\right) + \frac{\pi}{6}\right)\right) g_n(s)ds + \right.$$

$$\left. + B_{1,n}e^{-2\mu_n t} + B_{2,n}e^{\mu_n t}\cos\sqrt{3}\mu_n t + B_{3,n}e^{\mu_n t}\sin\sqrt{3}\mu_n t\right] v_n(x). \tag{2.14}$$

We find the unknown constants $B_{i,n}(t),\ \ i=1,2,3$ in this series (2.14) using the initial conditions (1.3). Since it is necessary for the following calculations, we also calculate $A''_{i,n}(t),\ \ i=1,2,3$:

$$A''_{1,n}(t) = \frac{1}{12\mu_n^2}\left[2\mu_n e^{2\mu_n t} g_n(t) + e^{2\mu_n t} g'_n(t)\right],$$

$$A''_{2,n}(t) = \frac{1}{6\mu_n^2}\left[\left(-\mu_n e^{-\mu_n t}\sin\left(\sqrt{3}\mu_n t + \frac{\pi}{6}\right) + \sqrt{3}\mu_n e^{-\mu_n t}\cos\left(\sqrt{3}\mu_n t + \frac{\pi}{6}\right)\right) g_n(t) + \right.$$

$$\left. + e^{-\mu_n t}\sin\left(\sqrt{3}\mu_n t + \frac{\pi}{6}\right) g'_n(t)\right],$$

$$A''_{3,n}(t) = -\frac{1}{6\mu_n^2}\left[\left(\mu_n e^{-\mu_n t}\cos\left(\sqrt{3}\mu_n t + \frac{\pi}{6}\right) + \sqrt{3}\mu_n e^{-\mu_n t}\sin\left(\sqrt{3}\mu_n t + \frac{\pi}{6}\right)\right) g_n(t) + \right.$$

$$\left. + e^{-\mu_n t}\cos\left(\sqrt{3}\mu_n t + \frac{\pi}{6}\right) g'_n(t)\right].$$

We also calculate each $u_{i,n}(0),\ \ u'_{i,n}(0),\ \ u''_{i,n}(0)$ and $A_{i,n}(0),\ \ A_{i,n}(0),\ \ A_{i,n}(0)\quad i=1,2,3$:

$$\begin{cases} u_{1,n}(0) = 1, \\ u_{2,n}(0) = 1, \\ u_{3,n}(0) = 0, \end{cases} \begin{cases} u'_{1,n}(0) = -2\mu_n, \\ u'_{2,n}(0) = \mu_n, \\ u'_{3,n}(0) = \sqrt{3}\mu_n, \end{cases} \begin{cases} u''_{1,n}(0) = 4\mu_n^2, \\ u''_{2,n}(0) = -2\mu_n^2, \\ u''_{3,n}(0) = 2\sqrt{3}\mu_n^2. \end{cases}$$

In that case, we will have:

$$\begin{cases} A_{1,n}(0) = B_{1,n}, \\ A_{2,n}(0) = B_{2,n}, \\ A_{3,n}(0) = B_{3,n}, \end{cases} \begin{cases} A'_{1,n}(0) = \frac{1}{12\mu_n^2} g_n(0), \\ A'_{2,n}(0) = -\frac{1}{12\mu_n^2} g_n(0), \\ A'_{3,n}(0) = \frac{\sqrt{3}}{12\mu_n^2} g_n(0), \end{cases}$$

$$
\begin{cases}
A''_{1,n}(0) = \frac{1}{12\mu_n^2}\left(2\mu_n g_n(0) + g'_n(0)\right), \\
A''_{2,n}(0) = -\frac{1}{12\mu_n^2}\left(2\mu_n g_n(0) + g'_n(0)\right), \\
A''_{3,n}(0) = \frac{\sqrt{3}}{12\mu_n^2}\left(2\sqrt{3}\mu_n g_n(0) - \sqrt{3} g'_n(0)\right).
\end{cases}
$$

Using the eigenfunctions $\vartheta_n(x)$, we expand the initial functions (1.3) into a Fourier series to obtain the following:

$$
u_n(T) = \int_0^1 U(T,x)\,\vartheta_n(x)dx = \int_0^1 \varphi_1(x)\,\vartheta_n(x)dx = \varphi_{1,n}, \tag{2.15}
$$

$$
u'_n(0) = \int_0^1 U_t(0,x)\,\vartheta_n(x)dx = \int_0^1 \varphi_2(x)\,\vartheta_n(x)dx = \varphi_{2,n}, \tag{2.16}
$$

$$
u_n(0) + u''_n(0)\int_0^1 \left(U(0,x) + U_{tt}(0,x)\right)\vartheta_n(x)dx = \int_0^1 \varphi_3(x)\,\vartheta_n(x)dx = \varphi_{3,n}. \tag{2.17}
$$

Taking into account the values, calculated above, and the formulas (2.15)–(2.17), we obtain the following expressions:

$$
u_n(t) = A_{1,n}(t)u_{1,n}(t) + A_{2,n}(t)u_{2,n}(t) + A_{3,n}(t)u_{3,n}(t) + B_{1,n}u_{1,n}(t) + B_{2,n}u_{2,n}(t) + B_{3,n}u_{3,n}(t). \tag{2.18}
$$

From the equation (2.18) we find the unknowns $B_{i,n}(t)$, $i = 1,2,3$. To do this, we consider the following equation:

$$
u_n(T) = A_{1,n}(T)u_{1,n}(T) + A_{2,n}(T)u_{2,n}(T) + A_{3,n}(T)u_{3,n}(T) + B_{1,n}u_{1,n}(T) + B_{2,n}u_{2,n}(T) +
$$

$$
+B_{3,n}u_{3,n}(T) = \alpha(T) + u_{1,n}(T)B_{1,n} + u_{2,n}(T)B_{2,n} + u_{3,n}(T)B_{3,n} = \varphi_{1,n}, \tag{2.19}
$$

where

$$
\alpha(T) = A_{1,n}(T)u_{1,n}(T) + A_{2,n}(T)u_{2,n}(T) + A_{3,n}(T)u_{3,n}(T),
$$

$$
u'_n(0) = A'_{1,n}(0)u_{1,n}(0) + A_{1,n}(0)u'_{1,n}(0) + A'_{2,n}(0)u_{2,n}(0) + A_{2,n}(0)u'_{2,n}(0) +
$$

$$
+A'_{3,n}(0)u_{3,n}(0) + A_{3,n}(0)u'_{3,n}(0) + B_{1,n}u'_{1,n}(0) + B_{2,n}u'_{2,n}(0) + B_{3,n}u'_{3,n}(0) =
$$

$$
= -2\mu_n B_{1,n} + \mu_n B_{2,n} + \sqrt{3}\mu_n B_{3,n} = \varphi_{2,n}. \tag{2.20}
$$

By similar way

$$
u_n(0) + u''_n(0) = \left(4\mu_n^2 + 1\right)B_{1,n} - \left(2\mu_n^2 - 1\right)B_{2,n} + 2\sqrt{3}\mu_n^2 B_{3,n} = \varphi_{3,n} \tag{2.21}
$$

and by the equations (2.19), (2.20) we obtain a system of equations for $B_{i,n}$ $(i = 1;2;3)$:

$$
\begin{cases}
u_{1,n}(T)B_{1,n} + u_{2,n}(T)B_{2,n} + u_{3,n}(T)B_{3,n} = \varphi_{1,n} - \alpha(T), \\
-2\mu_n B_{1,n} + \mu_n B_{2,n} + \sqrt{3}\mu_n B_{3,n} = \varphi_{2,n}, \\
(4\mu_n^2 + 1)B_{1,n} - (2\mu_n^2 - 1)B_{2,n} + 2\sqrt{3}\mu_n^2 B_{3,n} = \varphi_{3,n}.
\end{cases} \tag{2.22}
$$

We solve the system of linear equations (2.22) using Cramer's method:

$$
\Delta = \begin{vmatrix} u_{1,n}(T) & u_{2,n}(T) & u_{3,n}(T) \\ -2\mu_n & \mu_n & \sqrt{3}\mu_n \\ 4\mu_n^2 + 1 & 1 - 2\mu_n^2 & \sqrt{3}\mu_n^2 \end{vmatrix} = \sqrt{3}\mu_n(4\mu_n^2 - 1)u_{1,n}(T) +
$$

$$+\sqrt{3}\mu_n(8\mu_n^2+1)u_{2,n}(T)-3\mu_n u_{3,n}(T). \tag{2.23}$$

We will analyze the case where $\Delta=0$ in more depth. Using (2.7), we substitute the appropriate expressions for $u_{i,n}(T),\ \ i=1;2;3$ and obtain the equation $\Delta=\mu_n F(\mu_n)=0$, where

$$F(\mu_n)=\sqrt{3}(4\mu_n^2-1)e^{-2\mu_n T}+e^{\mu_n T}\left[\sqrt{3}\left(8\mu_n^2+1\right)\cos\left(\sqrt{3}\mu_n T\right)-3\sin\left(\sqrt{3}\mu_n T\right)\right]. \tag{2.24}$$

Since $\mu_n\neq 0$ Let's look at the equation $F(\mu_n)=0$. It is known that in $n\to\infty$, $\mu_n\to\frac{\omega}{2}$ so instead of equation (**??**) $F\left(\frac{\omega}{2}\right)=\sqrt{3}(\omega^2-1)e^{-\omega T}+e^{\frac{\omega}{2}T}\left[\sqrt{3}\left(2\omega^2+1\right)\cos\left(\sqrt{3}\frac{\omega}{2}T\right)-3\sin\left(\sqrt{3}\frac{\omega}{2}T\right)\right]=0$. Let's look at the equation, from this

$$\frac{\sqrt{3}(\omega^2-1)}{e^{1,5\omega T}}=3\sin\left(\sqrt{3}\frac{\omega}{2}T\right)-\sqrt{3}\left(2\omega^2+1\right)\cos\left(\sqrt{3}\frac{\omega}{2}T\right) \tag{2.25}$$

we will have equality.

The equations (2.25) have a finite number of solutions with respect to the parameter ω . If we denote the set of these solutions by Λ_1 , then we search for the solution of the problem (1.1)–(1.3) for the ω belonging to the set $\Lambda_2=(-\infty,0)\cup(0,\infty)/\Lambda_1$. To do this, we first calculate the corresponding determinants:

$$\Delta_{B_{1,n}}=\begin{vmatrix}\varphi_{1,n}-\alpha(T) & u_{2,n}(T) & u_{3,n}(T)\\ \varphi_{2,n} & \mu_n & \sqrt{3}\mu_n\\ \varphi_{3,n} & 1-2\mu_n^2 & 2\sqrt{3}\mu_n^2\end{vmatrix}=\sqrt{3}\mu_n(4\mu_n^2-1)(\varphi_{1,n}-\alpha(T))-$$

$$-\left(2\sqrt{3}\mu_n^2u_{2,n}(T)+(2\mu_n^2-1)u_{3,n}(T)\right)\varphi_{2,n}+\mu_n\left(\sqrt{3}u_{2,n}(T)-u_{3,n}(T)\right)\varphi_{3,n},$$

$$\Delta_{B_{2,n}}=\begin{vmatrix}u_{1,n}(T) & \varphi_{1,n}-\alpha(T) & u_{3,n}(T)\\ -2\mu_n & \varphi_{2,n} & \sqrt{3}\mu_n\\ 4\mu_n^2+1 & \varphi_{3,n} & 2\sqrt{3}\mu_n^2\end{vmatrix}=\sqrt{3}\mu_n(8\mu_n^2+1)(\varphi_{1,n}-\alpha(T))+$$

$$+\left(2\sqrt{3}\mu_n^2u_{1,n}(T)-(4\mu_n^2+1)u_{3,n}(T)\right)\varphi_{2,n}-\mu_n\left(\sqrt{3}u_{1,n}(T)+2u_{3,n}(T)\right)\varphi_{3,n},$$

$$\Delta_{B_{3,n}}=\begin{vmatrix}u_{1,n}(T) & \varphi_{1,n}-\alpha(T) & u_{3,n}(T)\\ -2\mu_n & \varphi_{2,n} & \sqrt{3}\mu_n\\ 4\mu_n^2+1 & \varphi_{3,n} & 2\sqrt{3}\mu_n^2\end{vmatrix}=-3\mu_n(\varphi_{1,n}-\alpha(T))+$$

$$+\left(\left(2\mu_n^2-1\right)u_{1,n}(T)+(4\mu_n^2+1)u_{2,n}(T)\right)\varphi_{2,n}+\mu_n\left(u_{1,n}(T)+2u_{2,n}(T)\right)\varphi_{3,n}.$$

Then, using Cramer's rule, we find the unknown quantities:

$$B_{1,n}=\frac{\sqrt{3}\mu_n(4\mu_n^2-1)}{\Delta}\varphi_{1,n}-\frac{2\sqrt{3}\mu_n^2u_{2,n}(T)+(2\mu_n^2-1)u_{3,n}(T)}{\Delta}\varphi_{2,n}+$$
$$+\frac{\mu_n\left(\sqrt{3}u_{2,n}(T)-u_{3,n}(T)\right)}{\Delta}\varphi_{3,n}-\frac{\sqrt{3}\alpha(T)\mu_n(4\mu_n^2-1)}{\Delta},$$

$$B_{2,n}=\frac{\sqrt{3}\mu_n(8\mu_n^2+1)}{\Delta}\varphi_{1,n}+\frac{2\sqrt{3}\mu_n^2u_{1,n}(T)-(4\mu_n^2+1)u_{3,n}(T)}{\Delta}\varphi_{2,n}-$$
$$-\frac{\mu_n\left(\sqrt{3}u_{1,n}(T)+2u_{3,n}(T)\right)}{\Delta}\varphi_{3,n}-\frac{\sqrt{3}\alpha(T)\mu_n(8\mu_n^2+1)}{\Delta},$$

$$B_{3,n}=-\frac{3\mu_n}{\Delta}\varphi_{1,n}+\frac{\left(2\mu_n^2-1\right)u_{1,n}(T)+(4\mu_n^2+1)u_{2,n}(T)}{\Delta}\varphi_{2,n}+$$
$$+\frac{\mu_n(u_{1,n}(T)+2u_{2,n}(T))}{\Delta}\varphi_{3,n}+\frac{3\mu_n\alpha(T)}{\Delta}.$$

The values can be written as follows:

$$B_{1,n}u_{1,n}(t) = \frac{\sqrt{3}\mu_n(4\mu_n^2-1)}{\Delta}u_{1,n}(t)\varphi_{1,n} - \frac{2\sqrt{3}\mu_n^2 u_{2,n}(T) + (2\mu_n^2-1)u_{3,n}(T)}{\Delta}u_{1,n}(t)\varphi_{2,n}+$$

$$+\frac{\mu_n\left(\sqrt{3}u_{2,n}(T) - u_{3,n}(T)\right)}{\Delta}u_{1,n}(t)\varphi_{3,n} - \frac{\sqrt{3}\alpha(T)\mu_n(4\mu_n^2-1)}{\Delta}u_{1,n}(t),$$

$$B_{2,n}u_{2,n}(t) = \frac{\sqrt{3}\mu_n(8\mu_n^2+1)}{\Delta}u_{2,n}(t)\varphi_{1,n} + \frac{2\sqrt{3}\mu_n^2 u_{1,n}(T) - (4\mu_n^2+1)u_{3,n}(T)}{\Delta}u_{2,n}(t)\varphi_{2,n}-$$

$$-\frac{\mu_n\left(\sqrt{3}u_{1,n}(T) + 2u_{3,n}(T)\right)}{\Delta}u_{2,n}(t)\varphi_{3,n} - \frac{\sqrt{3}\alpha(T)\mu_n(8\mu_n^2+1)}{\Delta}u_{2,n}(t),$$

$$B_{3,n}u_{3,n}(t) = -\frac{3\mu_n}{\Delta}u_{3,n}(t)\varphi_{1,n} + \frac{\left(2\mu_n^2-1\right)u_{1,n}(T) + (4\mu_n^2+1)u_{2,n}(T)}{\Delta}u_{3,n}(t)\varphi_{2,n}+$$

$$+\frac{\mu_n\left(u_{1,n}(T) + 2u_{2,n}(T)\right)}{\Delta}u_{3,n}(t)\varphi_{3,n} + \frac{3\mu_n\alpha(T)}{\Delta}u_{3,n}(t).$$

We write the above $B_{i,n},\ \ n=1,2,3$ by grouping them by $\varphi_{i,n},\ \ n=1,2,3$:

$$P_n(t,\omega) = B_{1,n}u_{1,n}(t) + B_{2,n}u_{2,n}(t) + B_{3,n}u_{3,n}(t) =$$

$$= \psi_{1,n}(t)\varphi_{1,n} + \psi_{2,n}(t)\varphi_{2,n} + \psi_{3,n}(t)\varphi_{3,n} + \Phi(t), \tag{2.26}$$

where

$$\psi_{1,n}(t) = \frac{\sqrt{3}\mu_n(4\mu_n^2-1)}{\Delta}e^{-2\mu_n t} + \frac{\sqrt{3}\mu_n(8\mu_n^2+1)}{\Delta}e^{\mu_n t}\cos\sqrt{3}\mu_n t - \frac{3\mu_n}{\Delta}e^{\mu_n t}\sin\sqrt{3}\mu_n t,$$

$$\psi_{2,n}(t) = \left(\frac{-\left(2\sqrt{3}\mu_n^2 u_{2,n}(T) + (2\mu_n^2-1)u_{3,n}(T)\right)}{\Delta}e^{-2\mu_n t} + \right.$$

$$+\frac{\left(2\sqrt{3}\mu_n^2 u_{1,n}(T) - (4\mu_n^2+1)u_{3,n}(T)\right)}{\Delta}e^{\mu_n t}\cos\sqrt{3}\mu_n t-$$

$$\left.-\frac{\left(\left(2\mu_n^2-1\right)u_{1,n}(T) + (4\mu_n^2+1)u_{2,n}(T)\right)}{\Delta}e^{\mu_n t}\sin\sqrt{3}\mu_n t\right),$$

$$\psi_{3,n}(t) = \left(\frac{\mu_n\left(\sqrt{3}u_{2,n}(T) - u_{3,n}(T)\right)}{\Delta}\ e^{-2\mu_n t} + \frac{\mu_n\left(\sqrt{3}u_{1,n}(T) + 2u_{3,n}(T)\right)}{\Delta}e^{\mu_n t}\cos\sqrt{3}\mu_n t-\right.$$

$$\left.-\frac{\mu_n\left(u_{1,n}(T) + 2u_{2,n}(T)\right)}{\Delta}e^{\mu_n t}\sin\sqrt{3}\mu_n t\right),$$

$$\Phi_n(t) = -\frac{\sqrt{3}\alpha(T)\mu_n(4\mu_n^2-1)}{\Delta}e^{-2\mu_n t} - \frac{\sqrt{3}\alpha(T)\mu_n(8\mu_n^2+1)}{\Delta}e^{\mu_n t}\cos\sqrt{3}\mu_n t+$$

$$+\frac{3\mu_n\alpha(T)}{\Delta}e^{\mu_n t}\sin\sqrt{3}\mu_n t.$$

We introduce the following notation

$$K_n(t-s,\omega) = \frac{1}{12}\left[\left(e^{-2\mu_n(\omega)(t-s)} - 2e^{\mu_n(\omega)(t-s)}\sin\left(\sqrt{3}\mu_n(\omega)\left(s-t\right) + \frac{\pi}{6}\right)\right)\right]. \tag{2.27}$$

Taking into account the definitions (2.5), (2.26) and (2.27), we write the solutions of the system of finite equations (2.4) in the form of the following function:

$$u_n(t,\omega) = P_n(t,\omega) + \frac{1}{\mu_n^2(\omega)} \int_0^t K_n(t-s,\omega)\, g_n(s)ds, \quad \omega \in \Lambda_2 \tag{2.28}$$

or by taking into account (2.5) and putting (2.1) in the row, we get the formal solution of the problem (1.1)–(1.3)

$$U(t,x,\omega) = \sum_{n=1}^{\infty} \vartheta_n(x) \left[P_n(t,\omega) + \frac{2}{\omega} \sqrt[3]{\frac{1}{(1+\lambda_n)^2 \lambda_n}} \int_0^t K_n(t-s,\omega)\, f_n(s)ds \right], \quad \omega \in \Lambda_2. \tag{2.29}$$

Theorem 2.1. *If there exists a solution of Problem* (1.1)–(1.3)*, then it is unique for regular values of the parameter ω from the set Λ_2.*

Proof. We consider the regular values of the parameter ω from the set Λ_2. Suppose that there exist two different solutions $U_1(t,x)$ and $U_2(t,x)$ to the problem (1.1)–(1.3). Then the difference $U(t,x) = U_1(t,x) - U_2(t,x)$ is a solution of the equation (1.1), satisfying the conditions (1.2)–(1.3) with functions $\varphi(x) \equiv 0$, $f(t,x) \equiv 0$. Then, it follows from formulas (2.2) and (2.28) that

$$u_n(t) = \int_0^1 U(t,y)\vartheta_n(y)dy \equiv 0.$$

From this, due to the completeness of the system of eigenfunctions in the space $L_2[0,1]$, it follows that $U(t,x) = 0$ is almost everywhere on $[0,1]$ for any $t \in [0,T]$. Since $U \in C(\bar{\Omega})$, it follows that $U(t,x) \equiv 0$ in $\bar{\Omega}$. The Theorem 2.1 is proved. □

Smoothness conditions. Let in the domain $[0,1]$ the functions $\varphi_i(x)$ and $f(t,x)$ satisfying the conditions

$$\varphi_i(x) \in C_x^3[0,1], \qquad \varphi_i(1) = \frac{d}{dx}\varphi_i(0) = 0, \quad i = 1,2,3,$$

$$f(t,x) \in C_{t,x}^{0,2}(\tilde{\Omega}), \qquad f(t,1) = \frac{d}{dx} f(t,0) = 0.$$

Then, we integrate by parts $\varphi_{i,n} = \int_0^1 \varphi_i(y)\vartheta_n(y)dy$, $i = 1,2,3$ three times and $f_n(t) = \int_0^1 f(t,y)\vartheta_n(y)dy$ two times on the variable x, respectively, and obtain

$$\varphi_{i,n} = \left(\frac{2}{\pi(1+2n)} \right)^3 \varphi_{i,n}''', \quad i = 1,2,3,$$

$$\varphi_{i,n}''' = \sqrt{2} \int_0^1 \frac{\partial^3 \varphi_i(y)}{\partial y^3} \sin \sqrt{\lambda_n} y\, dy, \quad i = 1,2,3,$$

$$f_n(t) = -\left(\frac{2}{\pi(1+2n)} \right)^2 f_n''(t), \quad f_n''(t) = \sqrt{2} \int_0^1 \frac{\partial^2 f(t,y)}{\partial y^2} \cos \sqrt{\lambda_n} y\ dy.$$

Moreover, from smoothness conditions we have

$$\| \vec{\varphi}_i''' \|_{\ell_2} \leq C_1 \left\| \frac{\partial^3 \varphi_i(x)}{\partial x^3} \right\|_{L_2[0,1]},$$

$$\left\| \vec{f}''(t) \right\|_{B_2} \leq C_2 \max_{0\leq t\leq T} \left\| \frac{\partial^2 f(t,x)}{\partial x^2} \right\|_{L_2[0,1]}, \quad 0 < C_i = const, \;\; i = 1, 2.$$

Theorem 2.2. *Let the smoothness conditions be satisfied. Then, for the regular values of the parameter ω from the set Λ_2 the series (2.29) continuously differentiable and convergences absolute and smooth.*

Proof. We consider the series (2.29) and

$$\frac{\partial^2}{\partial x^2} U(t,x,\omega) = \sum_{n=1}^{\infty} \left(\frac{\pi(1+2n)}{2} \right)^2 \vartheta_n(x) \left[\psi_{1,n}(t,\omega)\varphi_{1,n} + \psi_{2,n}(t,\omega)\varphi_{2,n} + \right.$$

$$\left. + \psi_{3,n}(t,\omega)\varphi_{3,n} + \Phi_n(t,\omega) + \frac{2}{\omega} \sqrt[3]{\frac{1}{(1+\lambda_n)^2\lambda_n}} \int_0^t K_n(t-s,\omega)\, f_n(s) ds \right], \tag{2.30}$$

$$\frac{\partial^5}{\partial t^3 \partial x^2} U(t,x,\omega) = \sum_{n=1}^{\infty} \left(\frac{\pi(1+2n)}{2} \right)^2 \vartheta_n(x) \left[\psi_{1,n}'''(t,\omega)\varphi_{1,n} + \psi_{2,n}'''(t,\omega)\varphi_{2,n} + \right.$$

$$\left. + \psi_{3,n}'''(t,\omega)\varphi_{3,n} + \Phi_n'''(t,\omega) + \frac{2}{\omega} \sqrt[3]{\frac{1}{(1+\lambda_n)^2\lambda_n}} \int_0^t K_n'''(t-s,\omega) f_n(s) ds \right], \tag{2.31}$$

where

$$\Phi_n(t,\omega) = -\frac{\sqrt{3}\alpha(T)\mu_n(\omega)(4\mu_n^2(\omega)-1)}{\Delta} e^{-2\mu_n(\omega)t} -$$

$$-\frac{\sqrt{3}\alpha(T)\mu_n(\omega)(8\mu_n^2(\omega)+1)}{\Delta} e^{\mu_n t} \cos\sqrt{3}\mu_n(\omega)t + \frac{3\mu_n(\omega)\alpha(T)}{\Delta_n(\omega)} e^{\mu_n t} \sin\sqrt{3}\mu_n(\omega)t,$$

$$K_n(t-s,\omega) = \frac{1}{12} \left[\left(e^{-2\mu_n(\omega)(t-s)} - 2e^{\mu_n(\omega)(t-s)} \sin\left(\sqrt{3}\mu_n(\omega)\,(s-t) + \frac{\pi}{6} \right) \right) \right].$$

We take into account that $0 < \mu_n(\omega) < \frac{\omega}{2}$ and the functions $\psi_{i,n}(t,\omega), \;\; i = 1,2,3, \;\; \Phi_n(t,\omega), \;\; K_n(t-s,\omega)$ are continuously differentiable many times, the proofs of convergence of the series (2.29)–(2.31) are similar. So, we will prove, for instance, of convergence for the series (2.30).

Appling the smoothness conditions to (2.30), we have:

$$\left| \frac{\partial^2}{\partial x^2} U(t,x,\omega) \right| \leq \sqrt{2} \sum_{n=1}^{\infty} \left(\frac{\pi(1+2n)}{2} \right)^2 \left[\| \psi_{1,n}(t,\omega) \|_C \, |\varphi_{1,n}| + \| \psi_{2,n}(t,\omega) \|_C \, |\varphi_{2,n}| + \right.$$

$$\left. + \| \psi_{3,n}(t,\omega) \|_C \, |\varphi_{3,n}| + \| \Phi_n(t,\omega) \|_C + \frac{2}{\omega} \sqrt[3]{\frac{1}{(1+\lambda_n)^2\lambda_n}} \int_0^t \| K_n(t-s,\omega) \|_C \; |f_n(s)|\, ds \right] \leq$$

$$\leq a_0 \left[\frac{2\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{1}{1+2n} |\varphi_{1,n}'''| + \frac{2\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{1}{1+2n} |\varphi_{2,n}'''| + \frac{2\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{1}{1+2n} |\varphi_{3,n}'''| + 1 \right] +$$

$$+\frac{2}{\omega}\sum_{n=1}^{\infty}\sqrt[3]{\frac{1}{(1+\lambda_n)^2\lambda_n^7}}\int\limits_0^t \|\, K_n(t-s,\omega)\,\|_C \,\left|\, f_n''(t)\,\right| ds \leq$$

$$\leq a_0\left[a_1\,\|\,\vec{\varphi}_1'''\,\|_{\ell_2}+a_1\,\|\,\vec{\varphi}_2'''\,\|_{\ell_2}+a_1\,\|\,\vec{\varphi}_3'''\,\|_{\ell_2}+1\right]+a_2\left\|\,\vec{f}'''(t)\,\right\|_{B_2}\int\limits_0^t \|\, K_n(t-s,\omega)\,\|_C \, ds,$$

where

$$a_0=\max\left\{\|\,\psi_{1,n}(t,\omega)\,\|_C\,,\ \|\,\psi_{2,n}(t,\omega)\,\|_C\,,\ \|\,\psi_{3,n}(t,\omega)\,\|_C\,,\ \|\,\Phi_n(t,\omega)\,\|_C\right\},$$

$$a_1=\frac{\sqrt{2}}{\pi}\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}=\sqrt{\frac{1}{3}},\quad a_2=\frac{2}{\omega}\sqrt{\sum_{n=1}^{\infty}\sqrt[3]{\frac{1}{(1+\lambda_n)^4\lambda_n^{14}}}}.$$

Further, using Bessel inequalities, we obtain

$$\left|\frac{\partial^2}{\partial x^2}U(t,x,\omega)\right| \leq$$

$$\leq a_0\left[a_1C_1\left\|\frac{\partial^3\,\varphi_1(x)}{\partial\,x^3}\right\|_{L_2[0,1]}+a_1C_1\left\|\frac{\partial^3\,\varphi_2(x)}{\partial\,x^3}\right\|_{L_2[0,1]}+a_1C_1\left\|\frac{\partial^3\,\varphi_3(x)}{\partial\,x^3}\right\|_{L_2[0,1]}+1\right]+$$

$$+a_2C_2\max_{0\leq t\leq T}\left\|\frac{\partial^2 f(t,x)}{\partial\,x^2}\right\|_{L_2[0,1]}\int\limits_0^t \|\, K_n(t-s,\omega)\,\|_C \, ds<\infty.$$

Theorem 2.2 is proved. □

Discussion

The results obtained in this paper illustrate how the solvability of a higher-order boundary value problem can be reduced to the interaction of three essentially different mathematical components: the spectral structure of the spatial operator, the solvability of the temporal ordinary differential equations, and the convergence properties of the resulting Fourier expansion.

The spatial spectral problem is comparatively simple but plays a fundamental role in the entire construction. The boundary conditions

$$\vartheta(1)=0,\qquad \vartheta'(0)=0$$

lead to an explicitly computable sequence of eigenvalues and to the cosine eigenfunctions

$$\vartheta_n(x)=\sqrt{2}\cos(\sqrt{\lambda_n}x).$$

Completeness and orthonormality of this system allow the partial differential equation to be decomposed into independent modal equations. This decomposition is particularly advantageous because it separates the dependence on the spatial variable from the more complicated third-order dynamics in time.

For every spatial mode, the original problem reduces to a third-order ordinary differential equation whose coefficients depend on the parameter ω. The fundamental solutions contain one exponentially decreasing term and two exponentially modulated trigonometric terms. This structure is reflected directly in the determinant arising from the temporal boundary conditions. Consequently, the spectral parameter does not enter the solvability theory only as a multiplicative constant in the original equation; it affects the invertibility of the finite-dimensional boundary system associated with every Fourier mode.

The analysis of the characteristic determinant is therefore one of the key points of the paper. The parameter values for which this determinant vanishes constitute an exceptional set. At such values the boundary conditions become algebraically degenerate for at least one modal equation, and the formulas derived by Cramer's rule are no longer applicable. By excluding these values and introducing the regular parameter set, the solution coefficients become well defined. This separation between regular and exceptional values provides a natural spectral interpretation of the solvability condition.

The uniqueness theorem is an important consequence of this construction. It does not require an independent energy estimate. Instead, uniqueness follows directly from the Fourier representation. For the difference of two possible solutions, the forcing and all boundary data vanish. Regularity of the parameter implies that the coefficients of the corresponding homogeneous modal problems vanish, and completeness of the eigenfunctions then gives the vanishing of the entire solution. This argument shows that the regularity condition imposed on ω is directly connected with uniqueness.

Another substantial part of the analysis concerns the transition from a formal solution to a classical solution. A Fourier representation obtained by separation of variables is initially only formal. Since the differential operator contains derivatives up to third order in t and second order in x, including the mixed derivative

$$\frac{\partial^5 U}{\partial t^3 \partial x^2},$$

termwise differentiation must be justified. The smoothness assumptions imposed on the data are specifically designed for this purpose.

In particular, three integrations by parts in the Fourier coefficients of the functions φ_i yield additional powers of the spatial eigenvalues in the denominator, while two integrations by parts provide the required decay for the Fourier coefficients of f. The resulting estimates are then combined with Bessel's inequality. This mechanism explains why the regularity assumptions are not merely technical: they provide precisely the coefficient decay necessary to compensate for the powers of λ_n introduced when the Fourier series is differentiated with respect to x.

Theorem 2.2 therefore complements the uniqueness result in an essential way. The first theorem addresses the algebraic and spectral obstruction associated with the parameter, whereas the second theorem addresses analytic convergence and smoothness. Taken together, the two results provide the main components needed for classical solvability: regularity of the parameter, uniqueness of the modal solution, and convergence of the Fourier representation together with the required derivatives.

The method also has a constructive character. Once a regular value of ω is fixed, the eigenvalues λ_n, the quantities $\mu_n(\omega)$, the characteristic determinant, and the functions entering the Fourier coefficients can all be calculated explicitly. The solution is therefore represented in a form suitable not only for theoretical analysis but also, at least in principle, for numerical truncation of the Fourier series.

At the same time, the present approach has several natural limitations. First, the theory is developed only for parameter values outside the exceptional set. The resonant case, in which the determinant vanishes, requires a separate analysis involving compatibility conditions and possibly nonuniqueness. Second, the coefficients of the differential equation are constant, which makes explicit separation of variables possible. Variable coefficients would generally destroy the simple modal structure and would require more sophisticated spectral or operator methods. Third, the rectangular geometry and the specific homogeneous spatial conditions are important for obtaining the explicit orthonormal basis used in the proof.

These limitations also suggest several directions for further research. Of particular interest is the solvability of the problem at exceptional parameter values, including Fredholm-type compatibility conditions and the structure of the corresponding solution space. One may also study the continuous dependence of the solution on ω, especially when the parameter approaches the exceptional set. Further extensions could include variable coefficients, nonhomogeneous or nonlocal spatial boundary conditions, higher-dimensional domains, inverse problems for the parameter, and numerical approximations based on truncated Fourier expansions.

From a broader perspective, the results confirm the usefulness of the spectral Fourier method for higher-order partial differential equations with mixed temporal and spatial differentiation. The combination of explicit spectral analysis, variation of parameters, finite-dimensional determinant conditions, and Fourier coefficient estimates provides a coherent framework in which both algebraic solvability and analytic regularity can be treated within the same construction.

Conclusion

A boundary value problem for a fifth-order partial differential equation with a real parameter has been investigated in a rectangular domain. The equation combines a third-order derivative with respect to time, a fifth-order mixed derivative with respect to time and space, and a second-order spatial term containing the parameter ω^3. The problem is supplemented by homogeneous boundary conditions in the spatial variable and by three nonstandard conditions involving values of the solution and its time derivatives at $t = 0$ and $t = T$.

The Fourier method reduces the spatial part of the problem to an explicitly solvable Sturm–Liouville problem. Its eigenfunctions form a complete orthonormal system in $L_2(0, 1)$, which makes it possible to expand the solution and the right-hand side in Fourier series. The corresponding Fourier coefficients satisfy a countable family of third-order ordinary differential equations. Using the method of variation of parameters, explicit representations of their solutions were derived.

The temporal boundary conditions lead to a three-dimensional algebraic system for each Fourier mode. Its characteristic determinant determines the admissible values of the parameter. The zeros of the corresponding characteristic expression form an exceptional set, while the complement defines the set of regular parameter values. For parameters belonging to this regular set, the algebraic systems are invertible and the Fourier coefficients of the solution are uniquely determined.

It has been shown that, for regular values of ω, the boundary value problem has at most one solution. The proof follows from the uniqueness of the Fourier coefficients and completeness of the spatial eigenfunction system. Thus, the parameter restriction is not merely a technical assumption but represents the spectral nonresonance condition underlying uniqueness.

Under additional smoothness and compatibility assumptions on the boundary functions and the right-hand side, the Fourier coefficients possess sufficient decay. Repeated integration by parts and Bessel-type estimates make it possible to establish absolute and uniform convergence of the Fourier series and of the derivatives required by the fifth-order differential operator. Consequently, the formal Fourier representation becomes a classical solution whenever the existence conditions are fulfilled.

The analysis therefore separates the solvability problem into two complementary parts. The first is spectral and algebraic: the parameter must avoid the exceptional values determined by the characteristic determinant. The second is analytic: the data must possess sufficient regularity to guarantee convergence and termwise differentiation of the Fourier series. This separation clarifies the mechanism by which the parameter and the smoothness of the data influence the boundary value problem.

The obtained results extend the Fourier analysis of higher-order partial differential equations with nonstandard temporal conditions and provide an explicit constructive representation of the solution. Further investigations may address the resonant parameter case, existence and compatibility conditions at exceptional values, stability with respect to the parameter, inverse problems, variable-coefficient equations, and numerical realization of the derived Fourier representation.

Declarations

Funding: There is no research Funding.

Data availability: This manuscript has no associated data.

Ethical Conduct: Not applicable.

Conflicts of interest: The authors declare that there is no conflict of interest.

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