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On the Hutchinson Population Model for Differential Equations with Maxima

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Abstract. *This paper investigates a generalized Hutchinson population model formulated as a nonlinear functional-differential equation involving a maximum operator. The proposed model extends the classical delayed logistic equation by incorporating the maximum population density attained over a recent time interval, thereby accounting for memory effects and delayed biological responses. Such formulations arise naturally in population dynamics, ecology, economics, and other systems characterized by hereditary behavior and feedback mechanisms.*
The analytical properties of the model are examined using the method of steps, which reduces the original nonlinear functional-differential equation to a sequence of ordinary differential equations on successive intervals. Explicit analytical solutions are derived for the initial stages of the construction, while numerical approximations are obtained using the Euler, Runge–Kutta, and Runge–Kutta–Merson methods. A comparison of the numerical schemes demonstrates the superior accuracy and stability of higher-order Runge–Kutta techniques.
In addition, a more general nonlinear boundary value problem containing a maximum functional and a product of nonlinear perturbation terms is considered. By transforming the problem into an equivalent nonlinear Fredholm integral equation and applying the method of successive approximations together with Banach's fixed-point theorem, sufficient conditions for the existence and uniqueness of solutions are established. The obtained results contribute to the qualitative theory of functional-differential equations with maxima and provide effective analytical and numerical tools for studying population models with memory.

Key words: *Hutchinson population model, functional–differential equations, existence and uniqueness, numerical methods, product of two nonlinear function.*

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Introduction

Mathematical models based on differential equations play a fundamental role in describing dynamic processes arising in biology, ecology, physics, engineering, economics, and the social sciences. Among these applications, population dynamics occupies a central position because it provides a quantitative framework for understanding growth, regulation, competition, and long-term stability of biological populations.

Classical population models, beginning with the works of Malthus and Verhulst, successfully describe unrestricted and density-dependent population growth. Later developments by Lotka and Volterra extended these ideas to interacting species and established the foundations of modern theoretical ecology. Nevertheless, many biological systems exhibit hereditary effects and delayed responses that cannot be adequately represented by ordinary differential equations. In particular, the influence of maturation time, reproductive age, and environmental memory often requires the incorporation of functional dependencies on past states of the population.

One of the most influential models accounting for such effects was proposed by G. E. Hutchinson, who introduced a delayed logistic equation in which the population growth rate depends on the population size at an earlier time. This model revealed the crucial role of delay parameters in determining the stability of equilibria and the emergence of oscillatory dynamics. Since then, delayed differential equations have become an important tool in ecological modeling and related disciplines.

In many real-world systems, however, the current state is influenced not only by a single delayed value but also by extreme characteristics of past behavior. For example, resource depletion, environmental pressure, predator response, and collective behavioral mechanisms may depend on the maximum population size attained during a recent period. Such considerations naturally lead to differential equations with maxima, which form an important subclass of functional-differential equations.

Motivated by these observations, this paper investigates a generalized Hutchinson-type model in which the delayed population term is replaced by the maximum value of the population over a prescribed interval. The resulting equation combines nonlinear dynamics, memory effects, and extremal functional dependence. The study focuses on both qualitative and numerical aspects of the model.

First, monotone solutions of the proposed equation are analyzed and conditions governing their existence are derived. Next, the method of steps is employed to construct solutions on successive intervals. This approach yields analytical representations for the first stages of the solution and provides a convenient framework for numerical implementation. Euler, Runge–Kutta, and Runge–Kutta–Merson schemes are then applied and compared.

Finally, a more general nonlinear model involving external perturbations represented by products of nonlinear functions is considered. By reducing the corresponding boundary value problem to an equivalent nonlinear Fredholm integral equation, existence and uniqueness results are obtained through the contraction mapping principle. The developed methodology extends classical techniques for delayed population models and contributes to the broader theory of functional-differential equations with maxima.

However, differential equations of mathematical physics have direct applications in the theory of nanosystems (see, for example [1]–[13] and [14]). Moreover, differential equations constitute a fundamental mathematical tool for modeling natural and social processes (see, for example, [15]–[24] and see also [25]– [32]). In population biology and ecology, mathematical models have played a central role since the classical works of Malthus [29], Quetelet [30], and Verhulst [31], which introduced the basic principles of demographic regulation. These early studies laid the groundwork for modern population dynamics.

In the first half of the twentieth century, the pioneering contributions of Lotka [28] and Volterra [32] to the theory of interacting species significantly advanced ecological modeling and led to the emergence of theoretical ecology as an independent scientific discipline. Despite extensive subsequent research, the mechanisms governing population regulation and stability remain a central topic of contemporary studies [33].

1 Problem statement

Classical logistic-type equations successfully describe population growth under simplified assumptions. However, such models do not account for maturation delays and age-structured effects. To address this limitation, G. E. Hutchinson proposed a delayed differential equation that explicitly incorporates the average reproductive age of a species. This model explains the appearance of oscillatory behavior and highlights the fundamental role of reproduction parameters. In partially, to describe the population dynamics of a species, it was proposed a mathematical model (see, [16, 22])

$$\dot{x}(t) + rx(t) + rx^2(t)) = 0, \quad t \geq 0, \tag{1.1}$$

where $x(t) = N(t) + K$, $N(t)$ is the population size of the species, K is the capacity of the habitat, r is the reproduction rate.

However, equation (1.1) describes well the dynamics of population growth of protozoan microorganisms. In 1948, G. E. Hutchinson proposed a model described by a differential equation with delay [24, 25]

$$\dot{x}(t) + rx(t-h) + rx(t-h)x(t) = 0, \quad t \geq 0, \tag{1.2}$$

where $x(t) = N(t) + K$, h characterizes the average reproductive age of the species.

Hutchinson used equation (1.2) to describe changes in the size of a herd of cows on a pasture. However, equation (1.2) has been used by other scientists to describe other processes (see, for examples, [22]). Let us consider equation (1.2) with the initial condition $x(t) = \varphi(t)$, $t \in [-h, 0]$, $x(0) = \varphi_0$. For small $r > 0$ the equation (1.2) has a stability domain. According to this model, asymptotic stability of the equilibrium position occurs when $N(t) \equiv K$, $t \geq 0$, $0 < rh < \pi/2$. It should be noted that periodic fluctuations appear in the mathematical model for $rh > \pi/2$. Consequently, in Hutchinson's model, the reproductive rate r and the average reproductive age h of the species play a fundamental role.

In the present work, we extend Hutchinson's approach by considering a functional-differential equation with maxima, where the population growth rate depends on the maximum population size over a recent time interval. Such formulations naturally arise in ecological systems with memory effects and delayed feedback. We analyze analytical properties of the proposed model and construct numerical solutions using standard time-stepping techniques.

So, we propose the following model to describe the population dynamics of the species

$$\dot{x}(t) + r \max_{\tau \in [t-\delta, t]} x(\tau) + r\, x(t) \max_{\tau \in [t-\delta, t]} x(\tau) = 0, \quad t \geq 0, \tag{1.3}$$

where $x(t) = N(t) + K$, τ is a variable value that provides the maximum number of species on the segment $[t-\delta, t]$, δ characterizes the size of the reproductive age of the species, r is reproduction rate.

This paper is further development of the work [34]–[36].

Boundary value problems for fractional order functional-differential equations are studied in [37], [38]. Theoretically, differential equations with maxima are considered, in partially, in [39]–[45].

2 Monotone solutions of the differential equation (1.3)

We check out of existence of monotone solutions of the differential equation (1.3). The increasing solutions of the differential equations (1.3) we are looking in the following standard form

$$x(t) = e^{\lambda t}, \quad 0 < \lambda = \text{const}.$$

Then from the nonlinear differential equation (1.3) we obtain

$$\lambda + r + r\, e^{\lambda t} = 0, \quad t \geq 0.$$

This equation we rewrite as

$$\lambda + r + r\,(N(t) + K) = 0, \quad t \geq 0.$$

Hence, we obtain that

$$N(t) = -\frac{\lambda + r}{r} - K, \quad t \geq 0,$$

where $N(t)$ is the population size of the species, K is the capacity of the habitat, $r > 0$ is the reproduction rate.

It is interesting that $-\frac{\lambda+r}{r} - K > 0$ or $rK + r < -\lambda$. Hence, we get the condition $\lambda < -r(K+1)$ for the increasing solution of the differential equation (1.3). The solution of the equation (1.3) depends from the $r > 0$. It is clear, that $\lim\limits_{r\to 0} x(t) = 0$. Since $r > 0,\ K > 0$, then the equation (1.3) has no monotone increasing solution.

The decreasing solutions of the differential equations (1.3) we are looking in the form

$$x(t) = e^{-\lambda t}, \quad 0 < \lambda = \text{const}.$$

Then from the equation (1.3) we obtain

$$-\lambda + re^{\lambda\delta} + re^{\lambda\delta}\, e^{-\lambda t} = 0, \quad t \geq 0.$$

According the our assumption this equation we rewrite as

$$-\lambda + re^{\lambda\delta} + re^{\lambda\delta}\,(N(t) + K) = 0, \quad t \geq 0.$$

From the last equation, we obtain that

$$N(t) = \frac{\lambda - re^{\lambda\delta}}{re^{\lambda\delta}} - K, \quad t \geq 0.$$

We consider the following estimate $\frac{\lambda - re^{\lambda\delta}}{re^{\lambda\delta}} - K > 0$ or $re^{\lambda\delta}K + re^{\lambda\delta} < \lambda$. Hence, we get the condition $0 < r < \lambda/e^{-\lambda\delta}(K+1)$ for the decreasing solution of the differential equation (1.3). The solution of the equation (1.3) depends from the $r \neq 0$. It is clear, that $\lim\limits_{r\to 0} x(t) = 0$. Consequently, for all $\lambda > re^{\lambda\delta}(K+1)$ the solutions of differential equation (1.3) are decreasing.

This condition is possible, for example, if $r = e^{-\lambda\delta}$. In this case, the population size $N(t)$ of the species is decreasing. So, for $r = e^{-\lambda\delta}$, the decreasing solution of the equation (3) has the form $x(t) = \exp\big\{ - (N(t) + K + 1)t\big\}$.

Thus, monotonicity of population size $N(t)$ change is true only in the case of population decline. An increasing population size $N(t)$ is associated with non-monotonicity (oscillatory). From the condition $\lambda > re^{\lambda\delta}(K+1)$ we obtain the condition for reproductive age of the species $\delta < \left(\frac{\lambda}{r(K+1)}\right)^{\frac{1}{\lambda}}$. Hence, we deduce that population size $N(t)$ is decreasing, when reproductive age δ of the species is smaller than $\left(\frac{\lambda}{r(K+1)}\right)^{\frac{1}{\lambda}}$.

3 Method of steps for solving equation (1.3)

The method of steps for solving nonlinear equation (1.3) is good that after application of this method the equation (1.3) is reduced to the linear equation. Further, one can solve the linear equation by analytical or numerical methods. To solve numerically linear equation, Runge–Kutta and Runge–Kutta–Merson methods give excellent results.

Initial value conditions have the forms

$$x(t) = \varphi(t), \quad t \in [-\delta, 0], \quad x(0) = \varphi(0) = \varphi_0.$$

We suppose that the solution at every point of the beginning every interval is continuous:

$$x^{+}((n-1)\delta) = x^{-}((n-1)\delta).$$

Without loss of generality, we set, $r = \delta = 1, \quad \varphi_0(t) = t$.

Zero step. On the initial interval $[-1, 0]$ the solution has the form

$$x(t) = \varphi(t) = \varphi_0(t) = t. \tag{3.1}$$

First step. On the interval $[0, 1]$ we have $\max\limits_{\tau\in[t-1,t]} x(\tau) = \max\limits_{\tau\in[t-1,t]} \varphi_0(\tau) = t$. We consider the equation (1.3) with condition (3.1): $x(t) = \varphi_0(t) = t$. Then we get

$$\dot{x}(t) = \max_{\tau\in[t-1,t]} \varphi_0(\tau)(1 + x(t)), \quad t \in [0, 1]. \tag{3.2}$$

Since $\varphi_0(0) = 0, \quad H_1(s) = \max\limits_{\tau\in[s-1,s]} \varphi_0(\tau) = \max\limits_{\tau\in[s-1,s]} \tau = s$, from (3.2) we obtain

$$x(t) = -\int_0^t s(1 + x(s))ds. \tag{3.3}$$

Analytical solution of the equation (3.3) is: $x(t) = \exp\left\{\frac{-t^2}{2}\right\} - 1$. So, on the interval $[0, 1]$ we have solution of the equation (1.3)

$$x(t) = \varphi_1(t) = \exp\left\{\frac{-t^2}{2}\right\} - 1, \quad t \in [0, 1]. \tag{3.4}$$

Second step. Function $x(t) = \varphi_1(t) = \exp\left\{\frac{-t^2}{2}\right\} - 1,\ t \in (0,1)$ is strictly decreasing. Therefore, on the segment $[t-1,t]$ the maximum is reached at the left point $\tau \in t-1$. We consider the condition $x(1) = \varphi_1(1) = \exp\left\{\frac{-1^2}{2}\right\} - 1 = \exp\left\{\frac{-1}{2}\right\} - 1$. Then, we have the equation

$$\dot{x}(t) = -\max_{\tau\in[t-1,t]} \varphi_1(\tau)(1+x(t)), \quad t \in [1,2]. \tag{3.5}$$

Since on the interval $[1,2]$ from the function (3.4) we have

$$H_2(t) = \max_{\tau\in[t-1,t]} x(\tau) =$$

$$= \varphi_1(t-1) = \max_{\tau\in[t-1,t]} \left[\exp\left\{\frac{-(t-1)^2}{2}\right\} - 1\right] = \exp\left\{\frac{-(t-1)^2}{2}\right\} - 1,$$

then from the differential equation (3.5) we obtain

$$x(t) = 1 - \int_1^t \left[\exp\left\{\frac{-(s-1)^2}{2}\right\} - 1\right](1+x(s))ds.$$

Separating variables in an equation (3.5), we obtain

$$\frac{d}{dt}\ln(1+x(t)) = -\left[\exp\left\{\frac{-(t-1)^2}{2}\right\} - 1\right].$$

We integrate from 1 till t:

$$\ln\frac{1+x(t)}{1+\ x(1)} = -\int_1^t \left[\exp\left\{\frac{-(s-1)^2}{2}\right\} - 1\right] ds. \tag{3.6}$$

Using substitution $u = s-1$, we get

$$\int_1^t \exp\left\{\frac{-(s-1)^2}{2}\right\} ds = \int_0^{t-1} \exp\left\{\frac{-u^2}{2}\right\} du = \sqrt{\frac{\pi}{2}}\, erf\left(\frac{t-1}{\sqrt{2}}\right). \tag{3.7}$$

Since $1+x(1.1) = 1+\varphi_1(1.1)$, then substituting (3.7) into (3.6) we obtain an analytical solution

$$x(t) = \varphi_2(t) = e^{-1/2}\exp\left\{t-1-\sqrt{\frac{\pi}{2}}\, erf\left(\frac{t-1}{\sqrt{2}}\right)\right\} - 1,\ \ t\in[1,2]. \tag{3.8}$$

So, the function (3.8) is an analytical solution of the equation (1.3) on the interval $[1,2]$.

Third step. Now we will solve the equation (1.3) on the interval $[2,3]$ From formula (3.8) we have that $x(2) = \varphi_2(2) = -0.299264$. So, on the interval $[2,3]$ we get

$$\max_{\tau\in[t-1,t]} x(\tau) = \max_{\tau\in[t-1,t]} \varphi_2(\tau) = e^{-1/2}\exp\left\{t-2-\sqrt{\frac{\pi}{2}}\, erf\left(\frac{t-2}{\sqrt{2}}\right)\right\} - 1.$$

Then, by integrating the equation

$$\dot{x}(t) = -\left\{e^{-1/2}\exp\left\{t-2-\sqrt{\frac{\pi}{2}} erf\left(\frac{t-2}{\sqrt{2}}\right)\right\} - 1\right\}(1+x(t)),\ \ t\in[2,3] \tag{3.9}$$

with the condition $x(2) = \varphi_2(2) = -0.299264$, we obtain

$$x(t) = \varphi_3(t) = -0.299264 - \int_2^t H_3(s)(1 + x(s))ds,$$

where

$$H_3(s) = \max_{\tau \in [s-1,s]} \varphi_2(\tau) = e^{-1/2} \exp\left\{s - 2 - \sqrt{\frac{\pi}{2}}\, erf\left(\frac{s-2}{\sqrt{2}}\right)\right\} - 1.$$

So, on the interval $[2, 3]$ as a solution of the equation (3.9) we have integral equation

$$x(t) = -0.299264-$$

$$-e^{-1/2} \int_2^t \left[\exp\left\{s - 2 - \sqrt{\frac{\pi}{2}}\, erf\left(\frac{s-2}{\sqrt{2}}\right)\right\} - 1\right](1 + x(s))ds, \ \ t \in [2, 3]. \tag{3.10}$$

The integral equation we solve numerically.

Table 1. Numerical results (step $h = 0.1$)

t	Euler	RK4	RK-Merson
2.000000	-0.299264	-0.299264	-0.299264
2.100000	-0.278294	-0.279037	-0.279037
2.200000	-0.258885	-0.260687	-0.260687
2.300000	-0.241639	-0.244854	-0.244854
2.400000	-0.227233	-0.232242	-0.232242
2.500000	-0.216443	-0.223795	-0.223795
2.600000	-0.209433	-0.219730	-0.219730
2.700000	-0.206453	-0.220566	-0.220566
2.800000	-0.207866	-0.229327	-0.229326
2.900000	-0.214181	-0.248392	-0.248392
3.000000	-0.243940	-0.266402	-0.266401

Table 1 shows that Euler method has the lowest accuracy. The fourth-order Runge–Kutta and Runge–Kutta–Merson methods yield similar and more stable results.

$k-$**step.** Continuing the process (3.4), (3.8) and (3.10) and integrating the following equation

$$\dot{x}(t) = -\max_{\tau \in [t-1,t]} \varphi_{k-1}(\tau)(1 + x(t)), \quad t \in [(k-1), k]$$

with condition $x((k-1)) = \varphi_{k-1}((k-1))$, we obtain

$$x(t) = \varphi_k(t) = \varphi_{k-1}((k-1)) - \int_{(k-1)}^t H_k(s)(1 + x(s))ds,$$

where $H_k(s) = \max\limits_{\tau \in [s-1,s]} \varphi_{k-1}(\tau)$.

Further, we can use the numerical methods:

1). Euler method: $h = 0.1$, $x_{n+1} = x_n + h \cdot f(t_n, x_n)$.

2). Runge–Kutta method:

$$k_{1n} = f(t_n, x_n), \quad k_{2n} = f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{1n}\right), \quad k_{3n} = f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{2n}\right),$$

$$k_{4n} = f\left(t_n + h, x_n + hk_{3n}\right), \qquad x_{n+1} = x_n + \frac{h}{6}\left(k_{1n} + 2k_{2n} + 2k_{3n} + k_{4n}\right).$$

3). Runge–Kutta–Merson method:

$$k_{1n} = hf(t_n, x_n), \quad k_{2n} = hf\left(t_n + \frac{h}{3}, x_n + \frac{k_{1n}}{3}\right), \quad k_{3n} = hf\left(t_n + \frac{h}{3}, x_n + \frac{k_{1n} + k_{2n}}{6}\right),$$

$$k_{4n} = hf\left(t_n + \frac{h}{2}, x_n + \frac{k_{1n} + 3k_{3n}}{8}\right), \quad k_{5n} = hf\left(t_n + h, x_n + \frac{k_{1n} - 3k_{3n} + 4k_{4n}}{8}\right),$$

$$x_{n+1} = x_n + \frac{1}{6}\left(k_{1n} + 4k_{4n} + k_{5n}\right).$$

4 Main result. Existence theorem

If in the model (1.3) we take into account the heterogeneous habitat, migration factors associated with the amount of available food, and the influence of predators, then the mathematical model takes on a more complex form

$$\dot{x}(t) + r \max_{\tau\in[t-h,t]} x(\tau) + rx(t) \max_{\tau\in[t-h,t]} x(\tau) = g(t, x(t))f(t, x(t)), \quad t \geq 0, \tag{4.1}$$

where g and f are the external disturbing influence on the population dynamics of a species.

We will consider equation (4.1) with the following conditions

$$x(t) = \varphi(t), \ \ t \in [-h, 0], \quad x(0) + x(T) = \varphi_0. \tag{4.2}$$

We integrate the initial-boundary value problem (4.1), (4.2) on the interval $(0, t)$:

$$x(t) = x(0)+$$

$$+\int_0^t \left[-r \max_{\tau\in[s-h,s]} x(\tau) - rx(s) \max_{\tau\in[s-h,s]} x(\tau) + g(s, x(s))f(s, x(s))\right] ds. \tag{4.3}$$

Taking into account condition (4.2), from the equation (4.3) we obtain

$$x(0) = \frac{\varphi_0}{2}-$$

$$-\frac{1}{2}\int_0^T \left[-r \max_{\tau\in[s-h,s]} x(\tau) - rx(s) \max_{\tau\in[s-h,s]} x(\tau) + g(s, x(s))f(s, x(s))\right] ds. \tag{4.4}$$

Substituting (4.4) into equation (4.3), we obtain the nonlinear Fredholm integral equation

$$x(t) = J(t; x(t)) \equiv \frac{\varphi_0}{2}+$$

$$+\frac{K_0}{2}\int\limits_0^T\left[-r\max_{\tau\in[s-h,s]}x(\tau)-rx(s)\max_{\tau\in[s-h,s]}x(\tau)+g(s,x(s))f(s,x(s))\right]ds, \tag{4.5}$$

where

$$K_0=\begin{cases}-\frac{1}{2}, & t\le s\le T,\\ \frac{1}{2}, & 0\le s<t.\end{cases}$$

We use a Banach space $C\big([0,T],\mathbb{R}\big)$, that contains a function $u(t)$, defined and continuous on a segment $[0,T]$, with the norm

$$\|\,u(t)\,\|_{C[0,T]}=\max_{0\le t\le T}|\,u(t)\,|\,.$$

Let us also consider the Banach space

$$BD\big([0,T],\mathbb{R}\big)=\big\{u:[0,T]\to\mathbb{R};\ u(t)\in C\big([0,T],\mathbb{R}\big)\big\}\,,$$

in which the function exists and is bounded by the norm

$$\|\,u(t)\,\|_{BD[0,T]}=\|\,u(t)\,\|_{C[0,T]}+h\,\|\,\dot{u}(t)\,\|_{C[0,T]}\,,$$

where $0<h=\text{const}$.

Lemma 4.1 ([43], [44])**.** *For the difference of two functions with maxima there holds the following estimate*

$$\left\|\max_{\tau\in[t-h,t]}x(\tau)-\max_{\tau\in[t-h,t]}y(\tau)\right\|_{C[0,T]}=\|\,x(t)-y(t)\,\|_{C[0,T]}+h\,\|\,\dot{x}(t)-\dot{y}(t)\,\|_{C[0,T]}\,.$$

Theorem 4.1. *Let be fulfilled the following conditions:*
1). $M_g=\|\,g(t,x)\,\|_{C[0,T]}\,,\quad M_f=\|\,f(t,x)\,\|_{C[0,T]}\,,\ \ 0<M_g,M_f=\text{const}$*;*
2). $|\,g(t,x_1)-g(t,x_2)\,|\le L_g\,|\,x_1-x_2\,|\,,\ \ 0<L_g=\text{const}$*;*
3). $|\,f(t,x_1)-f(t,x_2)\,|\le L_f\,|\,x_1-x_2\,|\,,\ \ 0<L_f=\text{const}$*;*
4). $\rho_i=\max\big\{\rho_{i1};h\rho_{i2}\big\}<1,\ \ i=1,2,\ ...,\ n-1.$

Then for the equation (4.5) *has a unique solution in the class* $BD\,([0,T],\mathbb{R})$*, which can be found by following iteration process:*

$$x_0(t)=\frac{\varphi_0}{2},\quad x_n(t)=J(t;x_{n-1}(t)),\ \ n=1,2,3,... \tag{4.6}$$

Proof. According to the conditions of the theorem, for the first approximation from (4.5) and (4.6) we have

$$\|\,x_1(t)\,\|\le\frac{|\,\varphi_0|}{2}+$$

$$+\frac{1}{4}\int\limits_0^T\left[|\,r\,|\left\|\max_{\tau\in[s-h,s]}x_0(\tau)\right\|+|\,r\,|\,\|\,x_0(s)\,\|\left\|\max_{\tau\in[s-h,s]}x_0(\tau)\right\|+\right.$$

$$\left.+\,\|\,g(s,x_0(s))\,\|\,\|\,f(s,x_0(s))\,\|\right]ds\le\frac{|\,\varphi_0|}{2}+$$

$$+\frac{T}{4}\left[|\,r\,|\max\left\{\|\,\varphi(t)\,\|\,,\frac{|\,\varphi_0|}{2}\right\}+|\,r\,|\,\frac{|\,\varphi_0|}{2}\max\left\{\|\,\varphi(t)\,\|\,,\frac{|\,\varphi_0|}{2}\right\}+M_gM_f\right]=$$

$$= \frac{|\varphi_0|}{2} + \rho_{01} < \infty, \tag{4.7}$$

where

$$\rho_{01} = \frac{T}{4}\left[|\,r\,|\max\left\{\|\,\varphi(t)\,\|\,,\frac{|\varphi_0|}{2}\right\} + |\,r\,|\,\frac{|\varphi_0|}{2}\max\left\{\|\,\varphi(t)\,\|\,,\frac{|\varphi_0|}{2}\right\} + M_g M_f\right].$$

Taking into account estimate (4.7) from (4.6) we have estimate

$$\|\,x_1(t) - x_0(t)\,\| \le \frac{1}{4}\int\limits_0^T \left[|\,r\,|\left\|\max_{\tau\in[s-h,s]} x_0(\tau)\right\| + \right.$$

$$\left. + |\,r\,|\,\|\,x_0(s)\,\|\left\|\max_{\tau\in[s-h,s]} x_0(\tau)\right\| + \|\,g(s,x_0(s))\,\|\,\|\,f(s,x_0(s))\,\|\right] ds \le \rho_{01}. \tag{4.8}$$

Analogously, from (4.1) we have

$$\|\,\dot{x}_1(t) - \dot{x}_0(t)\,\| \le |\,r\,|\left\|\max_{\tau\in[s-h,s]} x_0(\tau)\right\| +$$

$$+ |\,r\,|\,\|\,x_0(s)\,\|\left\|\max_{\tau\in[s-h,s]} x_0(\tau)\right\| + \|\,g(s,x_0(s))\,\|\,\|\,f(s,x_0(s))\,\| \le$$

$$\le |\,r\,|\max\left\{\|\,\varphi(t)\,\|\,,\frac{|\varphi_0|}{2}\right\} + |\,r\,|\,\frac{|\varphi_0|}{2}\max\left\{\|\,\varphi(t)\,\|\,,\frac{|\varphi_0|}{2}\right\} + M_g M_f = \rho_{02}. \tag{4.9}$$

From (4.8) and (4.9) we obtain that there holds the estimate

$$\|\,x_1(t) - x_0(t)\,\|_{BD[0,T]} \le \rho_{01} + h\,\rho_{02}.$$

For the second difference we have

$$\|\,x_2(t) - x_1(t)\,\| \le \frac{T\,|\,r\,|}{4}\left\|\max_{\tau\in[t-h,t]} x_1(\tau) - \max_{\tau\in[t-h,t]} x_0(\tau)\right\| +$$

$$+\frac{T\,|\,r\,|}{4}\,\|\,x_1(t)\,\|\left\|\max_{\tau\in[t-h,t]} x_1(\tau) - \max_{\tau\in[t-h,t]} x_0(\tau)\right\| +$$

$$+\frac{T\,|r|}{4}\,\|x_1(t) - x_0(t)\|\left\|\max_{\tau\in[t-h,t]} x_0(\tau)\right\| + \frac{T}{4}\,\|g(t,x_1(t))\|\,\|f(t,x_1(t)) - f(t,x_0(t))\| +$$

$$+\frac{T}{4}\,\|\,f(t,x_0(t))\,\|\,\|\,g(t,x_1(t)) - g(t,x_0(t))\,\|\,.$$

Applying lemma to the last inequality, we derive:

$$\|\,x_2(t) - x_1(t)\,\| \le \frac{T\,|\,r\,|}{4}\,(\|\,x_1(t) - x_0(t)\,\| + h\,\|\,\dot{x}_1(t) - \dot{x}_0(t)\,\|) +$$

$$+\frac{T\,|\,r\,|}{4}\left(\frac{|\varphi_0|}{2} + \rho_{01}\right)(\|\,x_1(t) - x_0(t)\,\| + h\,\|\,\dot{x}_1(t) - \dot{x}_0(t)\,\|) +$$

$$+\frac{T\,|\,r\,|}{4}\,\|\,x_1(t) - x_0(t)\,\|\,\frac{|\varphi_0|}{2} + \frac{T}{4}\,[M_g L_f + M_f L_g]\,\|\,x_1(t) - x_0(t)\,\|\,.$$

Hence, we obtain

$$\| x_2(t) - x_1(t) \| \leq \chi_{11} \| x_1(t) - x_0(t) \| + \chi_{12} h \| \dot{x}_1(t) - \dot{x}_0(t) \| , \tag{4.10}$$

where

$$\chi_{11} = \frac{T}{4}\left[|r| \left(1 + \left(\frac{|\varphi_0|}{2} + \rho_{01} \right) + \max\left\{ \| \varphi(t) \| , \frac{|\varphi_0|}{2} \right\} \right) + M_g L_f + M_f L_g \right],$$

$$\chi_{12} = \frac{T\,|r|}{4}\left(1 + \frac{|\varphi_0|}{2} + \rho_{01} \right).$$

Since $\chi_{11} \geq \chi_{12}$, the estimate (4.10) we rewrite as

$$\| x_2(t) - x_1(t) \|_{C[0,T]} \leq \rho_{11} \left(\| x_1(t) - x_0(t) \|_{C[0,T]} + h \| \dot{x}_1(t) - \dot{x}_0(t) \|_{C[0,T]} \right) \leq$$

$$\leq \rho_{11} \left(\rho_{01} + h\, \rho_{02} \right) , \tag{4.11}$$

where $\rho_{11} = \chi_{11}$.

Analogously we have

$$\| \dot{x}_2(t) - \dot{x}_1(t) \|_{C[0,T]} \leq |r| \left(\| x_1(t) - x_0(t) \| + h \| \dot{x}_1(t) - \dot{x}_0(t) \| \right) +$$

$$+ |r| \left(\frac{|\varphi_0|}{2} + \rho_{01} \right) \left(\| x_1(t) - x_0(t) \| + h \| \dot{x}_1(t) - \dot{x}_0(t) \| \right) +$$

$$+ |r| \| x_1(t) - x_0(t) \| \max\left\{ \| \varphi(t) \| , \frac{|\varphi_0|}{2} \right\} + [M_g L_f + M_f L_g] \| x_1(t) - x_0(t) \| \leq$$

$$\leq \chi_{21} \| x_1(t) - x_0(t) \| + \chi_{22} h \| \dot{x}_1(t) - \dot{x}_0(t) \| ,$$

$$\chi_{21} = |r| \left(1 + \left(\frac{|\varphi_0|}{2} + \rho_{01} \right) + \max\left\{ \| \varphi(t) \| , \frac{|\varphi_0|}{2} \right\} \right) + M_g L_f + M_f L_g,$$

$$\chi_{22} = |r| \left(1 + \frac{|\varphi_0|}{2} + \rho_{01} \right).$$

From last estimate we get

$$\| \dot{x}_2(t) - \dot{x}_1(t) \|_{C[0,T]} \leq \rho_{12} \left(\| x_1(t) - x_0(t) \|_{C[0,T]} + h \| \dot{x}_1(t) - \dot{x}_0(t) \|_{C[0,T]} \right) \leq$$

$$\leq \rho_{12} \left(\rho_{01} + h\, \rho_{02} \right) , \tag{4.12}$$

where $\rho_{12} = \chi_{21}$.

From the estimates (4.11) and (4.12) we obtain that there hold the estimates

$$\| x_2(t) - x_1(t) \|_{BD[0,T]} \leq \rho_1 \| x_1(t) - x_0(t) \|_{BD[0,T]} ,$$

where $\rho_1 = \max\{\rho_{11}; h\rho_{12}\} < 1$;

$$\| x_2(t) - x_1(t) \|_{BD[0,T]} \leq \rho_{11} \left(\rho_{01} + h\, \rho_{02} \right) + h\, \rho_{12} \left(\rho_{01} + h\, \rho_{02} \right) =$$

$$= \left(\rho_{11} + h\, \rho_{12} \right) \left(\rho_{01} + h\, \rho_{02} \right) . \tag{4.13}$$

We need in the following estimate

$$\| x_2(t) \| \leq \frac{|\varphi_0|}{2} + \frac{1}{4}\int\limits_0^T \left[|r| \left\| \max_{\tau\in[s-h,s]} x_1(\tau) \right\| + \right.$$

$$\left. + |r| \, \| x_1(s) \| \left\| \max_{\tau\in[s-h,s]} x_1(\tau) \right\| + \| g(s, x_1(s)) \| \, \| f(s, x_1(s)) \| \right] ds +$$

$$\leq \frac{|\varphi_0|}{2} + \frac{T}{4}\left[|r| \left(\max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + \rho_{01} \right) + \right.$$

$$\left. + |r| \left(\frac{|\varphi_0|}{2} + \rho_{01} \right) \left(\max\left\{ \| \varphi(t) \|, \frac{|\varphi_0|}{2} \right\} + \rho_{01} \right) + M_g M_f \right] =$$

$$= \frac{|\varphi_0|}{2} + \frac{T}{4}\left[|r| \left(\frac{|\varphi_0|}{2} + \rho_{01} \right) \left(1 + \frac{|\varphi_0|}{2} + \rho_{01} \right) + M_g M_f \right] < \infty. \tag{4.14}$$

Taking into account (4.14), analogously to (4.12) and (4.13) we consider estimating next difference

$$\| x_3(t) - x_2(t) \|_{C[0,T]} \leq \frac{T\,|r|}{4} \left\| \max_{\tau\in[t-h,t]} x_2(\tau) - \max_{\tau\in[t-h,t]} x_1(\tau) \right\| +$$

$$+ \frac{T\,|r|}{4} \| x_2(t) \| \left\| \max_{\tau\in[t-h,t]} x_2(\tau) - \max_{\tau\in[t-h,t]} x_1(\tau) \right\| + \frac{T\,|r|}{4} \times$$

$$\times \| x_2(t) - x_1(t) \| \left\| \max_{\tau\in[t-h,t]} x_1(\tau) \right\| + \frac{T}{4} \| g(t, x_2(t)) \| \, \| f(t, x_2(t)) - f(t, x_1(t)) \| +$$

$$+ \frac{T}{4} \| f(t, x_1(t)) \| \, \| g(t, x_2(t)) - g(t, x_1(t)) \| \leq$$

$$\leq \rho_{21} \left(\| x_2(t) - x_1(t) \|_{C[0,T]} + h \, \| \dot{x}_2(t) - \dot{x}_1(t) \|_{C[0,T]} \right), \tag{4.15}$$

where we used denotation

$$\rho_{21} = \frac{T}{4}\left[|r| \left(\| x_2(t) \| + \left\| \max_{\tau\in[t-h,t]} x_1(\tau) \right\| \right) + M_g L_f + M_f L_g \right].$$

Similarly, from (4.12) and (4.13) we obtain the estimate

$$\| \dot{x}_3(t) - \dot{x}_2(t) \|_{C[0,T]} \leq \rho_{22} \left(\| x_2(t) - x_1(t) \|_{C[0,T]} + h \, \| \dot{x}_2(t) - \dot{x}_1(t) \|_{C[0,T]} \right), \tag{4.16}$$

where

$$\rho_{22} = |r| \left(\| x_2(t) \| + \left\| \max_{\tau\in[t-h,t]} x_1(\tau) \right\| \right) + M_g L_f + M_f L_g.$$

From the estimates (4.15) and (4.16) we derive that there hold the estimates

$$\| x_3(t) - x_2(t) \|_{BD[0,T]} \leq \rho_2 \, \| x_2(t) - x_1(t) \|_{BD[0,T]},$$

where $\rho_2 = \max\{\rho_{21}; h\rho_{22}\} < 1$;

$$\| x_3(t) - x_2(t) \|_{BD[0,T]} \leq \rho_{21} \left(\rho_{11} + h\,\rho_{12} \right) \left(\rho_{01} + h\,\rho_{02} \right) + h\,\rho_{22} =$$

$$= (\rho_{21} + h\,\rho_{22})\,(\rho_{11} + h\,\rho_{12})\,(\rho_{01} + h\,\rho_{02})\,. \tag{4.17}$$

Continuing the process (4.17), we obtain an estimate for arbitrary n:

$$\|\,x_n(t) - x_{n-1}(t)\,\|_{BD[0,T]} \leq \rho_{n-1}\,\|\,x_{n-1}(t) - x_{n-2}(t)\,\|_{BD[0,T]}\,,$$

where $\rho_{n-1} = \max\{\rho_{(n-1)1}; h\rho_{(n-1)2}\} < 1$;

$$\|\,x_n(t) - x_{n-1}(t)\,\|_{BD[0,T]} \leq$$

$$\leq \left(\rho_{(n-1)1} + h\,\rho_{(n-1)2}\right) \cdots (\rho_{21} + h\,\rho_{22})\,(\rho_{11} + h\,\rho_{12})\,(\rho_{01} + h\,\rho_{02})\,, \tag{4.18}$$

where

$$\rho_{(n-1)1} = \frac{T}{4}\left[|\,r\,|\left(\|\,x_{n-1}(t)\,\| + \left\|\max_{\tau\in[t-h,t]} x_{n-2}(\tau)\right\|\right) + M_g L_f + M_f L_g\right],$$

$$\rho_{(n-1)2} = |\,r\,|\left(\|\,x_{n-1}(t)\,\| + \left\|\max_{\tau\in[t-h,t]} x_{n-2}(\tau)\right\|\right) + M_g L_f + M_f L_g.$$

From (4.18) we obtain the following estimate:

$$\|\,x_n(t) - x_{n-1}(t)\,\|_{BD[0,T]} \leq \rho_0\rho_1\rho_2 \cdot\, ... \,\cdot \rho_{n-2}\rho_{n-1}, \tag{4.19}$$

where $\rho_i = \max\left\{\rho_{i1}; h\rho_{i2}\right\} < 1,\ \ i = 1, 2,\, ...,\, n-1$.

The quantities r, h, L_f we choose such that from (4.19) implies

$$\lim_{n\to\infty} \|\,x_n(t) - x_{n-1}(t)\,\|_{BD[0,T]} = 0.$$

Hence, we deduce that the right-hand side of the equation (4.5) is contracting operator and the problem (4.1), (4.2) has a unique solution in the space $BD\,([0,T],\mathbb{R})$. □

5 Examples

Example 5.1. On the interval $[1,4]$ we consider the equation

$$x'(t) + 0.5\max_{\tau\in[t-1,t]} x(\tau) + 0.5x(t)\max_{\tau\in[t-1,t]} x(\tau) = 2t + (x(t) - 0.5t^2)x(t). \tag{5.1}$$

We will consider equation (5.1) with the following initial condition

$$x(t) = \varphi(t) = t + 1, \quad t \in [0,1], \quad x(1) = 2. \tag{5.2}$$

The function $x(t) = 1 + t^2$ is the exact solution of the problem (5.1), (5.2).

Example 5.2. On the segment $[0,3]$ consider the equation

$$x'(t) + 0.2\max_{\tau\in[t-1,t]} x(\tau) + 0.2x(t)\max_{\tau\in[t-1,t]} x(\tau) = \frac{0.2(t^2+1)}{x(t)-1}x^2(t). \tag{5.3}$$

We will consider equation (5.3) with the following initial condition

$$x(t) = \varphi(t) = \frac{1}{0.5t+2}, \quad t \in [-1,0], \quad x(0) = \varphi(0) = \frac{1}{2}. \tag{5.4}$$

The function $x(t) = \frac{1}{t+2}$ is the exact solution of the problem (5.3), (5.4).

Conclusion

In this paper, a generalized Hutchinson population model involving a maximum functional has been investigated. The proposed formulation extends the classical delayed logistic equation by allowing the population growth dynamics to depend on the largest population level attained within a recent time interval. Such a modification provides a natural mathematical framework for describing memory-dependent processes in biological and ecological systems.

The qualitative analysis of the model demonstrated that monotone increasing solutions do not occur under the considered assumptions, whereas monotone decreasing solutions exist under appropriate restrictions on the reproductive and model parameters. These results reveal the significant influence of memory length and reproduction intensity on the long-term behavior of the population.

To construct solutions, the method of steps was applied, reducing the original functional-differential equation to a sequence of ordinary differential equations on successive intervals. Explicit analytical solutions were obtained for the initial stages of the process, illustrating the effectiveness of the proposed approach. Numerical computations performed using Euler, Runge–Kutta, and Runge–Kutta–Merson schemes showed that higher-order methods provide considerably improved accuracy and stability when compared with the classical Euler method.

A generalized nonlinear model containing external perturbation terms was also studied. By transforming the corresponding boundary value problem into an equivalent nonlinear Fredholm integral equation and employing the method of successive approximations together with Banach's fixed-point theorem, sufficient conditions ensuring the existence and uniqueness of solutions were established.

The obtained results enrich the theory of functional-differential equations with maxima and provide new analytical and computational tools for the study of nonlinear population models with memory effects. Future research may focus on stability analysis, bifurcation phenomena, periodic solutions, stochastic extensions, fractional-order generalizations, and applications to ecological, economic, and engineering systems where extremal memory effects play a significant role.

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References

1. Blinova I. V., Grishanov E. N., Popov A. I., Popov I. Y., Smolkina M. O. On spin flip for electron scattering by several delta-potentials for 1D Hamiltonian with spin-orbit interaction. *Nanosystems: Phys. Chem. Math.* **14** (4), 413–417 (2023). https://doi.org/10.17586/2220-8054-2023-14-4-413-417

2. Deka H., Sarma J. A numerical investigation of modified Burgers' equation in dusty plasmas with non-thermal ions and trapped electrons. *Nanosystems: Phys. Chem. Math.* **14** (1), 5–12 (2023). https://doi.org/10.17586/2220-8054-2023-14-1-5-12

3. Dweik J., Farhat H., Younis J. The space charge model. A new analytical approximation solution of Poisson–Boltzmann equation: the extended homogeneous approximation. *Nanosystems: Phys. Chem. Math.* **14** (4), 428–437 (2023). https://doi.org/10.17586/2220-8054-2023-14-4-428-437
4. Fedorov E. G., Popov I. Yu. Analysis of the limiting behavior of a biological neurons system with delay. *J. Phys.: Conf. Ser.* **2086** (012109) (2021).
5. Fedorov E. G., Popov I. Yu. Hopf bifurcations in a network of Fitzhigh-Nagumo biological neurons. *International Journ. Nonlinear Sciences and Numerical Simulation* **2021**. https://doi.org/10.1515/ijnsns-2021-0188.
6. Fedorov E. G. Properties of an oriented ring of neurons with the Fitz-Hugh-Nagumo model. *Nanosystems: Phys. Chem. Math.* **2021**, *12*(5), 553–562.
7. Irgashev B. Yu. Boundary value problem for a degenerate equation with a Riemann–Liouville operator. *Nanosystems: Phys. Chem. Math.* **14** (5), 511–517 (2023). https://doi.org/10.17586/2220-8054-2023-14-5-511-517
8. Kuljanov U. N. On the spectrum of the two-particle Schrodinger operator with point potential: one dimensional case. *Nanosystems: Phys. Chem. Math.* **14** (5), 505–510 (2023). https://doi.org/10.17586/2220-8054-2023-14-5-505-510
9. Parkash C, Parke W. C., Singh P. Exact irregular solutions to radial Schrodinger equation for the case of hydrogen-like atoms. *Nanosystems: Phys. Chem. Math.* **14**(1), 28–43 (2023). https://doi.org/10.17586/2220-8054-2023-14-1-28-43
10. Popov I. Y. A model of charged particle on the flat Mobius strip in a magnetic field. *Nanosystems: Phys. Chem. Math.* **14** (4), 418–420 (2023). https://doi.org/10.17586/2220-8054-2023-14-4-418-420
11. Sibatov R. T., Svetukhin V. V. Subdiffusion kinetics of nanoprecipitate growth and destruction in solid solutions. *Theor. Math. Phys.* **183**, 846–859 (2015).
12. Vatutin A. D., Miroshnichenko G. P., Trifanov A. I. Master equation for correlators of normalordered field mode operators. *Nanosystems: Phys. Chem. Math.* **13**(6), 628–631 (2022). https://doi.org/10.17586/2220-8054-2022-13-6-628-631
13. Uchaikin V. V., Sibatov R. T. *Fractional kinetics in solids: Anomalous charge transport in semiconductors, dielectrics and nanosystems.* CRC Press, Boca Raton, FL, 2013.
14. Matrasulov J., Yusupov J. R., Saidov A. A. Fast forward evolution in heat equation: Tunable heat transport in adiabatic regime. *Nanosystems: Phys. Chem. Math.* **14** (4), 421–427 (2023). https://doi.org/10.17586/2220-8054-2023-14-4-421-427
15. Bellman R., Cook K. *Differential-difference equations.* Mir, Moscow, 1967. [in Russian]
16. V. Volyperra, *Mathematical theory of the struggle for existence.* Nauka, Moscow, 1976.
17. Dolgiy Yu. F. *Automatic control.* Ural State University, Sverdlovsk, 1988. [in Russian]
18. Day W. A. *Thermodynamics of simple media with memory.* Mir, Moscow, 1974. [in Russian]
19. Zubov V. I. *Analytical dynamics of the phone system.* Leningrad State University Publishing House, Leningrad, 1983. [in Russian]
20. Kolmanovsky V. B., Nosov V. R. *Stability and periodic regimes of controlled systems with aftereffect.* Nauka, Moscow, 1981. [in Russian]
21. A. V. Prasolov, *Dynamic models with delay and their applications in economics and engineering.* Lan Publishing House, St. Petersburg, 2010.
22. Yu. N. Svirezhev, D. O. Logofet, *Stability of biological communities.* Nauka, Moscow, 1978.
23. J. Hale, *Theory of functional differential equations.* Mir, Moscow, 1984.
24. Hessard B., Kazarinov H., Wen I. *Theory and applications of cycle birth bifurcation.* Mir, Moscow, 1985. [in Russian]

25. Z. Balanov, W. Krawcewicz, H. Ruan, "G. E. Hutchinson's delay logistic system with symmetries and spatial diffusion", *Nonlinear Analysis: Real World Applications* **9** (1), 154–182, (2008). https://doi.org/10.1016/j.nonrwa.2006.09.013

26. Shui-Nee Chow, John Mallet-Paret, "Integral averaging and bifurcation", *Journal of Differential Equations* **26** (1), 112–159 (1977), https://doi.org/10.1016/0022-0396(77)90101-2.

27. Shui-Nee Chow, John Mallet-Paret, "The Fuller index and global Hopf bifurcation", *Journal of Differential Equations* **29** (1), 66–85 (1978). https://doi.org/10.1016/0022-0396(78)90041-4.

28. A. Lotka, *Elements of physical biology.* Williams and Wilkins, Baltimore, 1925.

29. T. R. Malthus, *An essay on the principle of population.* J. Johnson, London, 1798.

30. A. Quetelet, *Sur l'homme et le d'eveloppement de ses facult'e.* Bachelier, Paris, 1835.

31. P. F. Verhulst, Notice sur la loi que la population suit dans son accroissement. *Correspondance Mathmatique Physique Publi'e A* **10**, 113–121 (1838).

32. V. Volterra, *Variations and fluctuations of the number of individuals in animal species living together. In Animal ecology (ed. R. N. Chapman)*, 409–448. McGraw Hill, New York, 1931.

33. A. Morozov, "Mathematical modelling in theoretical ecology: Introduction to the Special Issue", *Math. Model. Nat. Phenom.* **8** 6, 1–4 (2013). https://doi.org/10.1051/mmnp/20138601

34. A. Z. Pirmatov, R. N. Turaev, "On G. E. Hutchinson population model for fractional differential equations with maxima", *Journal of Osh University "Differential Equations"* **1** (1), 105–117 (2026).

35. T. K. Yuldashev, A. Pirmatov, R. Turaev, Sh. Alladustov, "Improving the G. E. Hutchinson population model for functional-differential equations", *Scientific Publications of the State University of Novi Pazar. Series A: Appl. Math. Inform. and Mechnics* **17** (2), 73–86 (2025). https://doi.org/10.46793/SPSUNP2502.073Y.

36. T. K. Yuldashev, R. N. Turaev, A. Z. Pirmatov, "On Hutchinson population model for differential equations with maxima and jumps", *Herald of Institute of Mathematics of NAS of KR* No. 2, 87–102 (2025).

37. A. Khalouta, "Existence and uniqueness of solution for Caputo–Fabrizio fractional Bratu-type initial value problem", *Azerbaijan Journal of Mathematics* **13** (1), 96–112 (2023).

38. N. H. Tuan, V. V. Tri, "Existence and continuous dependence of nonlocal final value problem with Caputo–Hadamard derivative", *Azerbaijan Journal of Mathematics* **15** (1), 144–168 (2025). https://doi.org/10.59849/2218-6816.2025.1.144

39. A. Molybaikyzy, A. D. Sarman, A. K. Tankeyeva, "Boundary value problem for a system of Fredholm integro-differential equations with maxima", *Journal of Osh University "Differential Equations"* **1** (1), 15–23 (2026).

40. T. K. Yuldashev, "On a nonlocal problem for impulsive differential equations with mixed maxima", *Vestnik KRAUNTS. Phys. and Math. Sci.* **38** (1), 40–53 (2022). https://doi.org/10.26117/2079-6641-2022-38-1-40-53

41. T. K. Yuldashev, T. A. Abduvahobov, "On a $(\omega, c)-$periodic solution for an impulsive system of second-order differential equation with product of two nonlinear functions and mixed maxima", *Uzbekistan Journal of Mathematics and Computer Science* **1** (1), 20–32 (2025). https://doi.org/10.56143/ujmcs.112032

42. T. K. Yuldashev, T. A. Abduvahobov, "Periodic solutions for an impulsive system of fractional order integro-differential equations with maxima", *Lobachevskii Journal of Mathematics* **44** (10), 4401–4409, (2023) https://doi.org/10.1134/S1995080223100451

43. A. K. Fayziyev, T. K. Yuldashev, "Periodic solutions of impulsive system of equations with a nonlinear function under the sign of a second-order differential and maxima", *Lobachevskii Journal of Mathematics* **46** (2), 658–671 (2025). https://doi.org/10.1134/S1995080225600256

44. T. K. Yuldashev, A. K. Fayziyev, "Inverse problem for a second order impulsive system of integro-differential equations with two redefinition vectors and mixed maxima", *Nanosystems: Physics. Chemistry. Mathematics* **14** (1), 13–21 (2023). https://doi.org/10.17586/2220-8054-2023-14-1-13-21

45. T. K. Yuldashev, Kh. R. Mamedov, T. A. Abduvahobov, "$(\omega, c)-$periodic solution for an impulsive system of differential equations with the quadrate of Gerasimov–Caputo fractional operator and maxima, *Journal of Contemporary Applied Mathematics* bf 13 (2), 65–79 (2023). http://doi.org/10.5281/zenodo.8425548