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Transformations of Hypergeometric Functions in Logarithmic Case with Applications to Study of Fundamental Solutions of Degenerate and Singular Elliptic Equations

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Abstract. Under some exceptional circumstances, such as when the sum of the upper numerical parameters of a hypergeometric function is equal to the sum of the lower parameters, or when these sums differ from each other by integers, a logarithmic function may be used in the expression of the hypergeometric function. In such cases we speak of logarithmic reducible hypergeometric functions and of logarithmic reduction formulas. In this paper, using the well-known logarithmic reduction formulas and statements about logarithmic behavior for the Gauss hypergeometric function and the second Appell function, we establish similar reduction formulas and statements for the first Lauricella function of three or more variables. The established theorems on logarithmic or power behavior for the single Gauss, double Appell and multiple Lauricella hypergeometric functions are applied to the determination of logarithmic or power singularities of fundamental solutions of degenerate and singular elliptic equations.

Key words: *Gauss, Appell and Lauricella hypergeometric functions; reduction formulas, logarithmic behavior, logarithmic singularity, degenerate and singular elliptic equations, fundamental solutions.*

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Introduction and preliminaries

The study of boundary value problems for degenerate equations is one of the important directions of the modern theory of partial differential equations. It is known that in the formulation and construction of local and nonlocal boundary value problems solutions, the main role is played by fundamental solutions.

Appell [1] has defined, in 1880, four double series, which are all analogous to Gauss' $F(a, b; c; x)$. Following the Appell's work, Lauricella [2] has defined, in 1893, four hypergeometric series $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$, which are natural generalizations of the classical Gauss hypergeometric function F and the Appell functions F_1–F_4 to the case of n complex variables and the corresponding complex parameters. The Appell and Lauricella hypergeometric functions play an important role in mathematical physics. Fundamental solutions of the two-dimensional degenerate elliptic equations are expressed by the Appell function F_2, and when the dimension of the equation exceeds two – by the Lauricella hypergeometric function $F_A^{(n)}$ with three and more variables. Under exceptional circumstances, multiple hypergeometric functions can be expressed in terms of simpler functions, notably in terms of hypergeometric functions of one variable or, in general, with fewer variables or in terms of elementary functions. In such cases we speak of *reducible* hypergeometric functions and of *reduction formulas.* If the sum of the upper numerical parameters of a hypergeometric function is equal to the sum of the lower parameters, or when these sums differ from each other by integers, then such case of the reducibility of hypergeometric functions we call *a logarithmic reduction formulas.*

The monographs [3]–[5] contain a systematic exposition of summation, reduction, and transformation formulas, as well as various other developments, covering not only each of the four Appell functions, but also several families of functions of two variables. Opps et al. [6] proved some reduction and transformation formulas for the Appell hypergeometric function F_2. Bezrodnykh [7] investigated an analytic continuation of the Appell function F_1 and integration of the associated system of equations in the logarithmic case. Furthermore, the works [8] and [9] compiled an extensive table of reduction formulas for the double Appell function F_2 and triple Lauricella function $F_A^{(3)}$, respectively.

Currently, the logarithmic reduction formulas and statements about logarithmic behavior for the Gauss hypergeometric function and the second Appell function are known. In this paper we establish similar reduction formulas and statements for the first Lauricella function of three or more variables and show an application of the established theorems on logarithmic or power behavior for the single Gauss, double Appell and multiple Lauricella hypergeometric functions to the determination of logarithmic or power singularities of fundamental solutions of degenerate and singular elliptic equations.

Below we give definition of Gamma function $\Gamma(z)$, Pochhammer symbol $(\kappa)_\nu$ and the logarithmic derivative of $\Gamma(z)$ which will be used in the next sections.

Gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \begin{cases} \int\limits_0^\infty t^{z-1} e^{-t} dt, \quad Re(z) > 0, \\ \\ \dfrac{\Gamma(z+1)}{z}, \quad Re(z) < 0; \quad z \neq -1, -2, -3, \ldots. \end{cases} \tag{0.1}$$

The definition (0.1) was used by Euler and there are other definitions of Gamma function (see, for instance, [4, p.1]).

A symbol $(\kappa)_\nu$ denotes the general Pochhammer symbol or the shifted factorial, since $(1)_l = l!$ $(l \in N \cup \{0\};\ N := \{1, 2, 3, ...\})$, which is defined (for $\kappa, \nu \in C$), in terms of the familiar Gamma function, by

$$(\kappa)_\nu := \frac{\Gamma(\kappa + \nu)}{\Gamma(\kappa)} = \begin{cases} 1 & (\nu = 0; \kappa \in C \backslash \{0\}) \\ \kappa(\kappa + 1)...(\kappa + l - 1) & (\nu = l \in N; \kappa \in C), \end{cases}$$

it being understood conventionally that $(0)_0 := 1$ and assumed tacitly that the $\Gamma-$ quotient exists.

The function $\psi(z)$ is the logarithmic derivative of the gamma function [4, p. 15]:

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \ln \Gamma(\mathrm{z}) = \int\limits_1^{\mathrm{z}} \psi(\mathrm{t}) \mathrm{dt},$$

possesses the following properties:

$$\psi(z) = \psi(1 + z) - \frac{1}{z}, \tag{0.2}$$

$$\psi(z + n) = \frac{1}{z} + \frac{1}{z + 1} + ... + \frac{1}{z + n - 1} + \psi(z), \quad n = 1, 2, 3, ... \tag{0.3}$$

1 Gaussian hypergeometric functions in logarithmic case

A function

$$F(a, b; c; z) \equiv F \left[\begin{array}{l} a, b; \\ c; \end{array} z \right] := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \ c \neq 0, -1, -2, ... \tag{1.1}$$

is known as the Gaussian hypergeometric function [4, p.56], where a, b, c are independent of z. We shall call a, b, c the parameters of the function (1.1); they are arbitrary complex numbers.

In study of Gaussian function (1.1), the meaning of the difference $c - a - b$ is very important. For example, if $Rec > Reb > 0$, $Re(c - a - b) > 0$, we have summation formula [4, p.61]

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.$$

If the sum of the upper parameters of a hypergeometric function differs from the similar sum of the lower parameters by an integer, then we speak of logarithmic reduction formulas or a logarithmic singularity.

If $c - a - b = 0$, we have for $c = a + b$ [10, p. 455, Eq. 7.3.1(30)]

$$F(a, b; a + b; z) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} [k_n - \ln(1 - z)] (1 - z)^n, \tag{1.2}$$

where

$$k_n = 2\psi(n + 1) - \psi(a + n) - \psi(b + n).$$

Putting $z \sim 1-\sigma$, one can rewrite a reduction formula (1.2) as follows

$$F(a,b;a+b;1-\sigma) = -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} F(a,b;1;\sigma)\ln\sigma + \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(n!)^2} k_n \sigma^n. \tag{1.3}$$

If $c-a-b$ is an integer, we have for $c = a+b+m$, $m = 0, 1, 2, \ldots$ [10, p. 455, Eq. 7.3.1(31)]

$$F(a,b;a+b+m;z) = \frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)} \sum_{n=0}^{m-1} \frac{(a)_n(b)_n}{(1-m)_n n!}(1-z)^n +$$

$$+(1-z)^m(-1)^m \frac{\Gamma(a+b+m)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a+m)_n(b+m)_n}{(m+n)!n!} \left[k'_n - \ln(1-z)\right](1-z)^n, \tag{1.4}$$

where

$$k'_n = \psi(n+1) + \psi(n+1+m) - \psi(a+n+m) - \psi(b+n+m)$$

and where $\sum\limits_{n=0}^{m-1}$ denotes zero if $m = 0$.

Finally, if $c = a+b-m$ where $m = 0, 1, 2, \ldots$, and if c is not an integer, we have [10, p. 455, 7.3.1(29)]

$$F(a,b;a+b-m;z) = (1-z)^{-m} \frac{\Gamma(m)\Gamma(a+b-m)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{m-1} \frac{(a-m)_n(b-m)_n}{(1-m)_n n!}(1-z)^n +$$

$$+(-1)^m \frac{\Gamma(a+b-m)}{\Gamma(a-m)\Gamma(b-m)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(m+n)!n!} \left[k''_n - \ln(1-z)\right](1-z)^n,$$

where

$$k''_n = \psi(n+1+m) + \psi(n+1) - \psi(a+n) - \psi(b+n)$$

and where $\sum\limits_{n=0}^{m-1}$ denotes zero if $m = 0$.

Examples. 1) Consider (1.2) for $a = 1$, $b = 1$. In this case

$$k_n := 2\psi(n+1) - \psi(1+n) - \psi(1+n) = 0.$$

Hence,

$$F(1,1;2;z) = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \sum_{n=0}^{\infty} \frac{(1)_n(1)_n}{(n!)^2} \left[0 - \ln(1-z)\right](1-z)^n = -\ln(1-z) \sum_{n=0}^{\infty} (1-z)^n,$$

i.e.

$$F(1,1;2;z) = -\frac{1}{z}\ln(1-z).$$

2) Consider (1.4) for $a = 1$, $b = 1$, $m = 1$. In this case $k'_n = \psi(n+1) - \psi(n+2)$ and by virtue of (0.2)

$$k'_n = -\frac{1}{n+1}. \tag{1.5}$$

Substituting (1.5) into (1.4), we have

$$F(1,1;3;z) = \frac{2}{z^2}\left[z + (1-z)\ln(1-z)\right].$$

In a similar way, using the properties (0.2) and (0.3) of the $\psi(z)$ function, one can create many examples (for details, see [10, Chap. 7]):

$$F\left(\frac{1}{4},1;\frac{5}{4};z\right) = \frac{1}{4x}\left[\ln\frac{1+x}{1-x} + 2\arctan x\right], \ x = z^{1/4};$$

$$F\left(\frac{1}{4},1;\frac{9}{4};z\right) = \frac{5}{16x^5}\left\{4x - \left(1-x^4\right)\left[\ln\frac{1+x}{1-x} + 2\arctan x\right]\right\}, \ x = z^{1/4};$$

$$F\left(\frac{3}{4},1;\frac{7}{4};z\right) = \frac{3}{4x^3}\left[\ln\frac{1+x}{1-x} - 2\arctan x\right], \ x = z^{1/4};$$

$$F\left(\frac{3}{4},1;\frac{11}{4};z\right) = \frac{7}{16x^7}\left\{4x^3 - 3\left(1-x^4\right)\left[\ln\frac{1+x}{1-x} - 2\arctan x\right]\right\}, \ x = z^{1/4};$$

$$F\left(1,2;3;z\right) = -\frac{2}{z^2}\left[z + \ln(1-z)\right];$$

$$F\left(1,3;4;z\right) = -\frac{3}{2z^3}\left[z^2 + 2z + 2\ln(1-z)\right]; F\left(2,3;4;z\right) = \frac{3}{z^3}\left[z + \frac{z}{1-z} + 2\ln(1-z)\right].$$

These reduction formulas can be proved without properties of the $\psi(z)$ function by comparing coefficients of equal powers of z on both sides.

2 Appell hypergeometric function F_2

The great success of the theory of hypergeometric function in one variable has stimulated the development of corresponding theory in two or more variables. Appell [1] has defined four functions F_1 to F_4, which are all analogues to Gauss' $F(a,b;c;x)$. For instance, the Appell function F_2 has a form [1]:

$$F_2\left(a,b_1,b_2;c_1,c_2;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\left(b_1\right)_m\left(b_2\right)_n}{\left(c_1\right)_m\left(c_2\right)_n}\frac{x^m}{m!}\frac{y^n}{n!}, \ \ |x|+|y|<1. \tag{2.1}$$

In the case of two variables, while investigating the behavior of fundamental solutions of a two-dimensional elliptic equation with two singular coefficients, E. T. Copson [11] determined the conditions under which the Appell hypergeometric function $F_2(a,b,b';c,c';x,y)$ has a logarithmic singularity as $x+y\to 1-0$.

Theorem 2.1 ([11]). *Let $x>0$, $y>0$. If $c>b>0$, $c'>b'>0$ and $a+b+b'=c+c'$, then*

$$F_2(a,b,b';c,c';x,y) \sim -\frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(b')}x^{b-c}y^{b'-c'}\ln(1-x-y)$$

as $x+y\to 1-0$. And, if $a+b+b'>c+c'$, then

$$F_2(a,b,b';c,c';x,y) \sim -\frac{\Gamma(c)\Gamma(c')\Gamma(a+b+b'-c-c')}{\Gamma(a)\Gamma(b)\Gamma(b')}x^{b-c}y^{b'-c'}(1-x-y)^{c+c'-a-b-b'}$$

as $x+y\to 1-0$.

Using some properties of Gaussian hypergeometric function $F(a,b;c;z)$, Opps et al. [6] proved the following theorem:

Theorem 2.2 ([6]). *For $|x|+|y|<1$, the Appell hypergeometric function F_2 is given by*

$$F_2\left(a+1,b,1;c,2;x,y\right)=-\frac{1}{ay}F\left(a,b;c;x\right)+\frac{1}{ay(1-y)^a}F\left(a,b;c;\frac{x}{1-y}\right),$$

where $a\neq 0,\ b\in\mathbb{C},\ c\in\mathbb{C}\setminus\mathbb{Z}_0$ and

$$F_2\left(1,b,1;c,2;x,y\right)=\frac{b}{cy}\left[\frac{x}{1-y}\ {}_3F_2\left(\begin{matrix}b+1,1,1;\\ c+1,2;\end{matrix}\ \frac{x}{1-y}\right)-\right.$$

$$\left.-x\ {}_3F_2\left(\begin{matrix}b+1,1,1;\\ c+1,2;\end{matrix}\ x\right)\right]-\frac{\ln(1-y)}{y},$$

where $b\in\mathbb{C},\ c\in\mathbb{C}\setminus\{0,-1,-2,...\}$; ${}_3F_2$ is a generalized hypergeometric function [4, Chap. 4]:

$${}_3F_2\left(\begin{matrix}b,b',b'';\\ c,c';\end{matrix}\ z\right)=\sum_{n=0}^{\infty}\frac{(b)_n\,(b')_n\,(b'')_n}{(c)_n\,(c')_n}\frac{z^n}{n!},\ \ |z|<1.$$

By applying this theorem, one can create examples of logarithmic reduction formulas.

Examples.

$$F_2\left(2,1,1;2,2;x,y\right)=\frac{1}{xy}\ln\frac{(1-x)(1-y)}{1-x-y};$$

$$F_2\left(3,1,1;3,2;x,y\right)=-\frac{1}{x^2y}\left[\frac{xy}{1-y}+\ln\frac{1-x-y}{(1-x)(1-y)}\right];$$

$$F_2\left(3,2,1;3,2;x,y\right)=\frac{1}{x(1-x)(1-x-y)}+\frac{1}{x^2y}\ln\frac{1-x-y}{(1-x)(1-y)};$$

$$F_2\left(1,\frac{1}{2},1;-\frac{1}{2},2;x,y\right)=\frac{1}{y}\left[\ln\frac{1-x}{1-x-y}-\frac{2xy}{(1-x)(1-x-y)}\right];$$

$$F_2\left(1,\frac{3}{2},1;\frac{1}{2},2;x,y\right)=\frac{1}{y}\left[\ln\frac{1-x}{1-x-y}+\frac{2xy}{(1-x)(1-x-y)}\right];$$

$$F_2\left(1,\frac{3}{2},1;-\frac{1}{2},2;x,y\right)=\frac{1}{y}\left[\ln\frac{1-x}{1-x-y}+\frac{4xy\left(x^2+y-1\right)}{(1-x)^2(1-x-y)^2}\right];$$

$$F_2\left(3,2,1;4,2;x,y\right)=\frac{3}{x^3y}\left[(2-x)\ln(1-x)+(x+2y-2)\ln\left(1-\frac{x}{1-y}\right)\right];$$

$$F_2\left(3,3,1;4,2;x,y\right)=\frac{3}{2x^3y}\left[\frac{x^2y}{(1-x)(1-x-y)}+2\left((1-y)\ln\frac{1-x-y}{1-y}-\ln(1-x)\right)\right];$$

$$F_2\left(1,b+1,1;b,2;x,y\right)=\frac{1}{b}\frac{x}{(1-x)(1-x-y)}+\frac{1}{y}\ln\frac{1-x}{1-x-y}.$$

These reduction formulas can be proved without the Theorem 2.2 by comparing coefficients of equal powers of x and y on both sides.

3 Lauricella hypergeometric function $F_A^{(n)}$

Lauricella [2] further generalized the four Applell series F_1, F_2, F_3, F_4 to series in n variables and defined his multiple hypergeometric series denoted by $F_A^{(n)}$, $F_B^{(n)}$,$F_C^{(n)}$ and $F_D^{(n)}$ which have important applications [12]–[14]. Particular values of Lauricella function $F_A^{(n)}$ are used to solve boundary value problems for the multidimensional Laplace equation in the hyperoctant of a multidimensional ball [15], [16]. New properties of Lauricella functions are found in [17]–[20]. In this paper all reduction formulas for the Appell and Lauricella hypergeometric functions are valid within the unit circle and the unit sphere, respectively. If the variables of these functions extend beyond the specified domains, then analytic continuations of the hypergeometric functions are used [21]. Confluent forms the multiple Lauricella hypergeometric functions have also important applications [22].

Lauricella hypergeometric function[2]

$$F_A^{(n)}\left(a,\mathbf{b};\mathbf{c};\mathbf{x}\right) \equiv F_A^{(n)}\left[\begin{array}{c} a, b_1, ..., b_n; \\ c_1, ..., c_n; \end{array} x_1, ..., x_n\right] = \sum_{k_1,...,k_n=0}^{\infty} (a)_{k_1+...+k_n} \prod_{i=1}^{n} \frac{(b_i)_{k_i}}{(c_i)_{k_i}} \frac{x_i^{k_i}}{k_i!}, \tag{3.1}$$

where $|x_1| + ... + |x_n| < 1$, is a natural generalization of the classical Gauss hypergeometric function (1.1) and the Appell function (2.1) to the case of many complex variables and their corresponding complex parameters. Hereinafter

$$\mathbf{b} := (b_1, ..., b_n)\,,\ \mathbf{c} := (c_1, ..., c_n)\,,\ \mathbf{x} := (x_1, ..., x_n)\,.$$

Lauricella hypergeometric function $F_A^{(n)}$ has the following integral representation [3, p. 115, Eq. (5)]:

$$F_A^{(n)}\left(a,\,\mathbf{b};\mathbf{c};\mathbf{x}\right) = \prod_{j=1}^{n}\left[\frac{\Gamma\left(c_j\right)}{\Gamma\left(b_j\right)\Gamma\left(c_j-b_j\right)}\right]\int_0^1 ... \int_0^1 \prod_{j=1}^{n}\left[\xi_j^{b_j-1}\left(1-\xi_j\right)^{c_j-b_j-1}\right] \times$$

$$\times\left(1-\sum_{j=1}^{n}\xi_j x_j\right)^{-a} d\xi_1 ... d\xi_n,\ \ Re\left(b_j\right)>0,\ \ Re\left(c_j-b_j\right)>0,\ j=\overline{1,n}. \tag{3.2}$$

The sums of the numerical parameters of the numerator and denominator will be denoted by $|\mathbf{b}|$ and $|\mathbf{c}|$, respectively, and by X we will denote the sum of the positive variables of Lauricella function $F_A^{(n)}$:

$$|\mathbf{b}| = b_1 + ... + b_n,\ \ |\mathbf{c}| = c_1 + ... + c_n,\ \ X = x_1 + ... + x_n,\ \ x_1 \geq 0, ..., x_n \geq 0.$$

It is known, that the Lauricella function $F_A^{(n)}\left(a,\mathbf{b};\ \mathbf{c};\ \mathbf{x}\right)$, defined in (3.1), converges absolutely if $x_1 > 0$, ..., $x_n > 0$, and $X < 1$. Clearly, the point M with coordinates $x_j = a_j^2/A$ lies on the hyperplane $X = 1$, where $A := a_1^2 + ... + a_n^2,\ \ a_1 > 0,\ ...\ a_n > 0$. We show that every point of this hyperplane is a point of logarithmic singularity, if $c_j > b_j > 0$ and $a + |\mathbf{b}| = |\mathbf{c}|$ $(j = \overline{1,n})$.

Let us denote the function in (3.1) by $F_A^{(n)}$ for brevity. Then, if $b_i > 0,\ \ i = \overline{1,n}$, then the integral representation (3.2) for the function $F_A^{(n)}$ can easily be transformed to the form

$$\prod_{j=1}^{n}\frac{\Gamma\left(b_j\right)\Gamma\left(c_j-b_j\right)}{\Gamma\left(c_j\right)}\,F_A^{(n)} = \int_0^1 ... \int_0^1 \left(1-\sum_{j=1}^{n}t_j x_j\right)^{-a}\prod_{j=1}^{n}\left[t_j^{b_j-1}\left(1-t_j\right)^{c_j-b_j-1}dt_j\right] = I,$$

where integration is over the hypercube $0 < t_1 < 1$, ..., $0 < t_n < 1$. The multiple integral is convergent when $x_1 > 0$, ..., $x_n > 0$, $X < 1$. We now write $x_j = \xi_j r$, where $\xi_j > 0$, $\xi_1 + ... + \xi_n = 1$, $0 < r < 1$, and consider the behavior of I as $r \to 1 - 0$. We have

$$I = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \int_0^1 ... \int_0^1 (t_1 x_1 + ... + t_n x_n)^k \prod_{j=1}^{n} \left[t_j^{b_j - 1} (1 - t_j)^{c_j - b_j - 1} \, dt_j \right] = a_0 + \sum_{k=1}^{\infty} \frac{(a)_k}{k!} E_k r^k,$$

where

$$a_0 = \prod_{j=1}^{n} \left[\frac{\Gamma(b_j) \Gamma(c_j - b_j)}{\Gamma(c_j)} \right]$$

and

$$E_k = \int_0^1 ... \int_0^1 (t_1 x_1 + ... + t_n x_n)^k \prod_{j=1}^{n} \left[t_j^{b_j - 1} (1 - t_j)^{c_j - b_j - 1} \, dt_j \right] =$$

$$= \int_0^1 ... \int_0^1 (1 - t_1 x_1 - ... - t_n x_n)^k \prod_{j=1}^{n} \left[t_j^{c_j - b_j - 1} (1 - t_j)^{b_j - 1} \, dt_j \right].$$

Now set $t_1 = \theta_1 / (k \xi_1)$, ... , $t_n = \theta_n / (k \xi_n)$. Then, by virtue of $|\mathbf{c}| - |\mathbf{b}| = a$, we obtain

$$E_k = \frac{1}{k^{|c| - |b|}} \prod_{j=1}^{n} \frac{1}{\xi_j^{c_j - b_j}} \int_0^{k\xi_1} ... \int_0^{k\xi_n} \left(1 - \frac{\theta_1 + ... + \theta_n}{k} \right)^k \prod_{j=1}^{n} \left[\theta_j^{c_j - b_j - 1} \left(1 - \frac{\theta_j}{k \xi_j} \right)^{b_j - 1} d\theta_j \right].$$

By virtue of the following (easily-derivable) identity

$$\lim_{k \to \infty} \int_0^k \left(1 - \frac{x}{n} \right)^k x^{\alpha - 1} dx = \int_0^{\infty} e^{-x} x^{\alpha - 1} dx,$$

we get

$$k^a E_k \to \prod_{j=1}^{n} \frac{1}{\xi_j^{c_j - b_j}} \int_0^{\infty} ... \int_0^{\infty} e^{-\Theta} \prod_{j=1}^{n} \left[\theta_j^{c_j - b_j - 1} d\theta_j \right] =$$

$$= \prod_{j=1}^{n} \left[\frac{1}{\xi_j^{c_j - b_j}} \int_0^{\infty} e^{-\theta_j} \theta_j^{c_j - b_j - 1} d\theta_j \right] = \prod_{j=1}^{n} \left[\frac{\Gamma(c_j - b_j)}{\xi_j^{c_j - b_j}} \right]$$

as $k \to \infty$. Using the well-known asymptotic formula for the gamma function [4, p.62, Eq. (4)]

$$\frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} = z^{\alpha - \beta} \left[1 + \frac{1}{2z} (\alpha - \beta)(\alpha + \beta - 1) + \mathrm{O}\left(\frac{1}{z^2} \right) \right], \; z \to \infty,$$

we obtain

$$\frac{(a)_k}{k!} E_k = \frac{E_k}{\Gamma(a)} \frac{\Gamma(a + k)}{\Gamma(1 + k)} \sim \frac{E_k}{\Gamma(a)} k^{a-1} =$$

$$= \frac{k^{a-1}}{\Gamma(a) k^a} \prod_{j=1}^{n} \left[\frac{\Gamma(c_j - b_j)}{\xi_j^{c_j - b_j}} \right] = \frac{1}{k \Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma(c_j - b_j)}{\xi_j^{c_j - b_j}} \right]$$

as $k \to \infty$.

Further, the following lemma is very important:

Lemma 3.1 ([23]). *If the power series* $\sum\limits_{n=0}^{\infty} a_n x^n$ *and* $\sum\limits_{n=0}^{\infty} b_n x^n$ *both converge within the interval* $(-1,1)$, a_n *being positive and such that the number series* $\sum\limits_{n=0}^{\infty} a_n$ *is divergent, and if* b_n/a_n *oscillates between the limits* U *and* L, *then the upper and lower limits of* $\sum\limits_{n=0}^{\infty} b_n x^n / \sum\limits_{n=0}^{\infty} a_n x^n$, *as* x *converges to* 0, *are in the interval* (L,U). *In particular, if* b_n/a_n *converges to a definite limit, as* $n \to \infty$, *then the expression* $\sum\limits_{n=0}^{\infty} b_n x^n / \sum\limits_{n=0}^{\infty} a_n x^n$ *converges to the same limit, as* x *converges to 1. If* b_n/a_n *diverges to* ∞, *as* $n \to \infty$, *so also does* $\sum\limits_{n=0}^{\infty} b_n x^n / \sum\limits_{n=0}^{\infty} a_n x^n$, *as* $x \ \sim 1$.

Using a Lemma 3.1, we have

$$\lim_{r \to 1-0} \frac{\sum\limits_{k=1}^{\infty} \frac{(a)_k}{k!} E_k r^k}{\sum\limits_{k=1}^{\infty} \frac{1}{k} r^k} = \lim_{k \to \infty} \frac{\frac{(a)_k}{k!} E_k}{\frac{1}{k}} = \frac{1}{\Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma(c_j - b_j)}{\xi_j^{c_j - b_j}} \right].$$

Hence, taking into account the well-known expansion of the logarithmic function

$$\ln(1-r) = -\sum_{k=1}^{\infty} \frac{1}{k} r^k,$$

we get

$$I \sim -\frac{1}{\Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma(c_j - b_j)}{\xi_j^{c_j - b_j}} \right] \ln(1-r) = -\frac{1}{\Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma(c_j - b_j)\, r^{c_j - b_j}}{x_j^{c_j - b_j}} \right] \ln(1-X) \sim$$

$$\sim -\frac{1}{\Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma(c_j - b_j)}{x_j^{c_j - b_j}} \right] \ln(1-X)$$

as $r \to 1-0$.

It follows that, if $c_j > b_j > 0$, $a + |\mathbf{b}| = |\,\mathbf{c}|$ and x_j are positive $(j = \overline{1,n})$, then

$$F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) \sim -\frac{1}{\Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma(c_j)}{\Gamma(b_j)} x_j^{b_j - c_j} \right] \cdot \ln(1-X); \tag{3.3}$$

as $X \to 1-0$. And, if $a + |\mathbf{b}| > |\mathbf{c}|$, then

$$F_A^{(n)}(a, \mathbf{b}; \mathbf{c}; \mathbf{x}) \sim \frac{\Gamma(a + |\mathbf{b}| - |\mathbf{c}|)}{\Gamma(a)} \prod_{j=1}^{n} \left[\frac{\Gamma(c_j)}{\Gamma(b_j)} x_j^{b_j - c_j} \right] \cdot (1-X)^{|\mathbf{c}| - |\mathbf{b}| - a}. \tag{3.4}$$

Thus, the following theorem is proved:

Theorem 3.1. *Let* $x_1 > 0, \ldots, x_n > 0$. *If* $c_1 > b_1 > 0$, ..., $c_n > b_n > 0$ *and* $a + |\mathbf{b}| = |\mathbf{c}|$, *then as* $X \to 1-0$ *the Lauricella function* $F_A^{(n)}$ *has a logarithmic singularity in the form* (3.3); *if* $c_1 > b_1 > 0$, ..., $c_n > b_n > 0$ *and* $a + |\mathbf{b}| > |\mathbf{c}|$, *then the power singularity* (3.4) *is valid.*

Theorem 3.1 in case $n = 2$ for Appell's function F_2 is proved in [11].

Note, the reduction formulas for the triple Lauricella function $F_A^{(3)}$ are established in [9].

4 Applications

A. A famous Tricomi equation

$$y^m \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \;\; m > 0 \tag{4.1}$$

in a half-plane $y \geq 0$ has two fundamental solutions:

$$q_1 (x, y; x_0, y_0) = k_1 r_1^{-2\beta} F(\beta, \beta; 2\beta; 1-\sigma), q_2 (x, y; x_0, y_0) = k_2 r_1^{-2\beta} \sigma^{1-2\beta} F(1-\beta, 1-\beta; 2-2\beta; 1-\sigma),$$

where k_1 and k_2 are known constants;

$$\left.\begin{matrix} r^2 \\ r_1^2 \end{matrix}\right\} = (x - x_0)^2 + \frac{4}{(m+2)^2} \left(y^{\frac{m+2}{2}} - x_0^{\frac{m+2}{2}} \right)^2; \;\; \sigma = \frac{r^2}{r_1^2}; \;\; \beta = \frac{m}{2(m+2)}.$$

These functions in variables x and y are solutions of the Tricomi equation (4.1), and by virtue of formula (1.3) it follows that for $r \to 0$ and $\sigma \to 0$ $(y > 0)$ they have a logarithmic singularity, and therefore, are fundamental solutions of the Tricomi equation (4.1).

B. The biaxially-symmetric potential equation

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{2\alpha}{x} \frac{\partial W}{\partial x} + \frac{2\beta}{y} \frac{\partial W}{\partial y} = 0$$

has the self-adjoint form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\alpha(1-\alpha)}{x^2} u + \frac{\beta(1-\beta)}{y^2} u = 0. \tag{4.2}$$

We wish to find a fundamental solution with singularity at (a, b) where a and b are positive.

We try to find a solution of (4.2) in the form

$$u = \xi^\alpha \eta^\beta v(\xi, \eta)$$

where

$$\xi = \frac{4ax}{R^2}, \;\; \eta = \frac{4by}{R^2}, \;\;\; R^2 = (x+a)^2 + (y+b)^2.$$

Since

$$1 - \xi = \frac{(x-a)^2 + (y+b)^2}{R^2}, \;\; 1 - \eta = \frac{(x+a)^2 + (y-b)^2}{R^2},$$

we have

$$0 < \xi < 1, \;\;\; 0 < \eta < 1$$

in the first quadrant. A straight-forward calculations shows that u is a solution if v satisfies the simultaneous equations

$$\begin{cases} \xi(1-\xi) v_{\xi\xi} - \xi\eta v_{\xi\eta} + [2\alpha - (2\alpha + \beta + 1)\, \xi]\, v_\xi - \alpha\eta v_\eta - \alpha(\alpha+\beta) v = 0, \\ \eta(1-\eta) v_{\eta\eta} - \xi\eta v_{\xi\eta} + [2\beta - (\alpha + 2\beta + 1)\eta]\, v_\eta - \beta\xi v_\xi - \beta(\alpha+\beta) v = 0. \end{cases}$$

These are the partial differential equations satisfied by Appell's function [4, p. 234]

$$v = F_2(\alpha + \beta, \alpha, \beta; 2\alpha, 2\beta; \xi, \eta).$$

Hence we have found a solution

$$u = \left(\frac{4ax}{R^2}\right)^{\alpha}\left(\frac{4by}{R^2}\right)^{\beta} F_2\left(\alpha+\beta, \alpha, \beta; 2\alpha, 2\beta; \frac{4ax}{R^2}, \frac{4by}{R^2}\right). \tag{4.3}$$

The double series

$$F_2(\alpha+\beta, \alpha, \beta; 2\alpha, 2\beta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha+\beta)_{m+n}(\alpha)_m(\beta)_n}{(2\alpha)_m(2\beta)_n m! n!}$$

is absolutely convergent when $x > 0$, $y > 0$, $x + y < 1$. The singularity of the proposed fundamental solution (4.3) corresponds in the present notation to

$$x = \frac{a^2}{a^2+b^2}, \quad y = \frac{b^2}{a^2+b^2}$$

which lies on the line $x+y = 1$. We show that every point of this line is a logarithmic singularity.

It follows from Theorem 3.1 that, if x and y are positive and $\alpha > 0$, $\beta > 0$,

$$F_2(\alpha+\beta, \alpha, \beta; 2\alpha, 2\beta; x, y) \sim -\frac{\Gamma(2\alpha)\Gamma(2\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)} x^{-\alpha} y^{-\beta} \ln(1-x-y) \tag{4.4}$$

as $x + y \to 1 - 0$. Substituting (4.4) into (4.3), we get

$$u \sim -\frac{\Gamma(2\alpha)\Gamma(2\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)} \ln\left(1 - \frac{4ax}{R^2} - \frac{4by}{R^2}\right) = -\frac{\Gamma(2\alpha)\Gamma(2\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)} \ln\frac{r^2}{R^2},$$

where $r^2 = (x-a)^2 + (y-b)^2$.

Hence,

$$u \sim \frac{\Gamma(2\alpha)\Gamma(2\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)} \ln\frac{1}{r}$$

as $r \to 0$. Thus, the fundamental solution defined in (4.3) has a logarithmic singularity as $r \to 0$.

C. The three-dimensional degenerate elliptic equation

$$y^m z^k u_{xx} + x^n z^k u_{yy} + x^n y^m u_{zz} = 0, m > 0, n > 0, k > 0 \tag{4.5}$$

in the domain $\Omega = \{(x,y,z) : x > 0,\ y > 0,\ z > 0\}$ has 8 fundamental solutions (for details, see [12]). We show that every fundamental solution of equation (4.5) has a singularity of order $\frac{1}{r}$ (non-logarithmic singularity) as $r \to 0$, where

$$r^2 = \frac{1}{q^2}\left(x^q - \xi^q\right)^2 + \frac{1}{p^2}\left(y^p - \eta^p\right)^2 + \frac{1}{l^2}\left(z^l - \zeta^l\right)^2, \ q = \frac{n+2}{2}, \ p = \frac{m+2}{2}, \ l = \frac{k+2}{2}.$$

To give an example, we consider one (first) fundamental solution:

$$q\left(x, y, z; \xi, \eta, \zeta\right) =$$

$$= \frac{\kappa}{r^{1+2\alpha+2\beta+2\gamma}} F_A^{(3)}\left[\begin{matrix} 1/2+\alpha+\beta+\gamma, \alpha, \beta, \gamma; \\ 2\alpha, 2\beta, 2\gamma; \end{matrix} \ -\frac{4x^q\xi^q}{q^2r^2}, -\frac{4y^p\eta^p}{p^2r^2}, -\frac{4z^l\zeta^l}{l^2r^2}\right], \tag{4.6}$$

where

$$\alpha = \frac{n}{2\,(n+2)}, \quad \beta = \frac{m}{2\,(m+2)}, \quad \gamma = \frac{k}{2\,(k+2)};$$

$$\kappa = \frac{1}{2\pi} q^{-2\alpha} p^{-2\beta} l^{-2\gamma} \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(2\alpha+2\beta+2\gamma)}{\Gamma(2\alpha)\Gamma(2\beta)\Gamma(2\gamma)\Gamma(\alpha+\beta+\gamma)}.$$

It is obvious that

$$0 < 2\alpha < 1, \;\; 0 < 2\beta < 1, \;\; 0 < 2\gamma < 1; \quad q > 1, \;\; p > 1, \;\; l > 1.$$

Using the transformation formula [3, p.116]

$$F_A^{(3)} \left[\begin{matrix} a, b_1, b_2, b_3; \\ c_1, c_2, c_3; \end{matrix} \; x, y, z \right] = (1 - x - y - z)^{-a} \times$$

$$\times F_A^{(3)} \left[\begin{matrix} a, c_1 - b_1, c_2 - b_2, c_3 - b_3; \\ c_1, c_2, c_3; \end{matrix} \; \frac{x}{x+y+z-1}, \frac{y}{x+y+z-1}, \frac{z}{x+y+z-1} \right],$$

the function q defined in (4.6) can be reduced to the form

$$q\,(x, y, z; \xi, \eta, \zeta) = \frac{\kappa}{R^{1+2\alpha+2\beta+2\gamma}} F_A^{(3)} \left[\begin{matrix} \alpha+\beta+\gamma+1/2, \alpha, \beta, \gamma; \\ 2\alpha, \;\; 2\beta, \;\; 2\gamma; \end{matrix} \; \frac{4x^q\xi^q}{q^2R^2}, \frac{4y^p\eta^p}{p^2R^2}, \frac{4z^l\zeta^l}{l^2R^2} \right],$$

where

$$R^2 = \frac{1}{q^2}\,(x^q + \xi^q)^2 + \frac{1}{p^2}\,(y^p + \eta^p)^2 + \frac{1}{l^2}\left(z^l + \zeta^l\right)^2.$$

By virtue of

$$0 < \frac{4x^q\xi^q}{q^2R^2} + \frac{4y^p\eta^p}{p^2R^2} + \frac{4z^l\zeta^l}{l^2R^2} < 1 \quad \text{and} \quad 1 - \frac{4x^q\xi^q}{q^2R^2} - \frac{4y^p\eta^p}{p^2R^2} - \frac{4z^l\zeta^l}{l^2R^2} = \frac{r^2}{R^2},$$

and applying the Theorem 3.1 (see, Eq. (3.4)), we obtain

$$q\,(x, y, z; \xi, \eta, \zeta) \sim \frac{1}{4\pi}(x\xi)^{-n/4}(y\eta)^{-m/4}(z\zeta)^{-k/4} \cdot \frac{1}{r}$$

as $r \to 0$, where $x > 0, y > 0, z > 0, \xi > 0, \eta > 0, \zeta > 0$.

The order of singularity of the remaining fundamental solutions of equation (4.5) is determined similarly. Thus, all fundamental solutions of the three-dimensional degenerate elliptic equation (4.5) have a singularity of order $\dfrac{1}{r}$ as $r \to 0$.

D. Consider a multidimensional elliptic equation with several singular coefficients

$$L_{(\alpha)}^m(u) \equiv \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} + \sum_{j=1}^{n} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} = 0 \tag{4.7}$$

in the domain $R_m^{n+} := \{(x_1, ..., x_m) : x_1 > 0, ..., x_n > 0\}$, where m is a dimension of the Euclidean space, n is a number of the singular coefficients of equation (4.7); $m \geq 2, 0 \leq n \leq m$; α_j are real constants and $0 < 2\alpha_j < 1$, $j = 1, ..., n$; $(\alpha) := (\alpha_1, ..., \alpha_n)$.

Let $x := (x_1, ..., x_m)$ be any point and $\xi := (\xi_1, ..., \xi_m)$ be any fixed point of R_m^{n+}. We are interested in case $m > 2$, $0 < n \leq m$. In this case, equation (4.7) has 2^n linearly independent solutions [24], which are expressed in terms of the Lauricella function $F_A^{(n)}$ defined in (3.1). One (the first) of these solutions is represented in the form

$$q(x;\xi)=\kappa r^{-2\beta}F_A^{(n)}\left[\begin{matrix}\beta,\alpha_1,...,\alpha_n;\\2\alpha_1,...,2\alpha_n;\end{matrix}\ -\frac{4x_1\xi_1}{r^2},...,-\frac{4x_n\xi_n}{r^2}\right], \tag{4.8}$$

$$r^2=\sum_{i=1}^{m}(x_i-\xi_i)^2;\ \ \beta:=\frac{m-2}{2}+\sum_{j=1}^{n}\alpha_j,\ \ \kappa:=2^{2\beta-m}\frac{\Gamma(\beta)}{\pi^{m/2}}\prod_{j=1}^{n}\frac{\Gamma(\alpha_j)}{\Gamma(2\alpha_j)}.$$

Using the transformation formula [3, p.116]

$$F_A^{(n)}\left[\begin{matrix}a,b_1,...,b_n;\\c_1,...,c_n;\end{matrix}\ x_1,...,x_n\right]=(1-x_1-...-x_n)^{-a}\times$$

$$\times F_A^{(n)}\left[\begin{matrix}a,c_1-b_1,...,c_n-b_n;\\c_1,...,c_n;\end{matrix}\ \frac{x_1}{x_1+...+x_n-1},...,\frac{x_n}{x_1+...+x_n-1}\right],$$

the function $q(x;\xi)$ defined in (4.8) can be reduced to the form

$$q(x;\xi)=\frac{\kappa}{R^{2\beta}}F_A^{(n)}\left[\begin{matrix}\beta,\alpha_1,...,\alpha_n;\\2\alpha_1,...,2\alpha_n;\end{matrix}\ \frac{4x_1\xi_1}{R^2},...,\frac{4x_n\xi_n}{R^2},\right],$$

where

$$R^2:=\sum_{j=1}^{n}(x_j+\xi_j)^2+\sum_{j=n+1}^{m}(x_j-\xi_j)^2.$$

By virtue of

$$0<4\sum_{j=1}^{n}x_j\xi_j<R^2\quad\text{and}\quad R^2-4\sum_{j=1}^{n}x_j\xi_j=r^2,$$

and applying the Theorem 3.1 (see Eq. (3.4)), we obtain

$$q(x;\xi)\sim\frac{1}{4\pi^{m/2}}\Gamma\left(\frac{m-2}{2}\right)\prod_{j=1}^{n}\left(\frac{x_j}{\xi_j}\right)^{\alpha_j}\cdot\frac{1}{r^{m-2}},\ m>2$$

as $r\to0$, where $x_1>0,...,x_n>0,\ \ \xi_1>0,...,\xi_n>0$.

Applications of fundamental solutions of the multidimensional elliptic equation with several singular coefficients (4.7) to solve a boundary value problems in the domain R_m^{n+} are found in [25].

Conclusion

Previously, several works were devoted to determining the type of singularity of fundamental solutions for degenerate or singular elliptic equations [26], [27]; however, these works used an infinite expansion of the Lauricella function in Gaussian hypergeometric functions in each variable and required a large number of transformations associated with the calculation of the values of the Gaussian functions for each variable. Theorem 3.1 allows us to quickly determine the type and order of singularity; for this, information about the sums of the upper and lower numerical parameters of the hypergeometric function is sufficient.

In the paper [28] all 2^n linearly independent solutions of the singular Helmholtz equation

$$\sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} + \sum_{j=1}^{n} \frac{2\alpha_j}{x_j} \frac{\partial u}{\partial x_j} + \lambda u = 0, \ -\infty < \lambda < \infty$$

in the domain R_+^n are constructed in explicit forms which are expressed through the confluent hypergeometric function in $n+1$ variables:

$$\mathrm{H}_A^{(n,1)} \left[\begin{array}{l} a, b_1, ..., b_n; \\ c_1, ..., c_n; \end{array} x_1, ..., x_n; y \right] = \sum_{k_1,...,k_n,m=0}^{\infty} (a)_{k_1+...+k_n-m} \prod_{i=1}^{n} \frac{(b_i)_{k_i}}{(c_i)_{k_i}} \frac{x_i^{k_i}}{k_i!} \frac{y^m}{m!}, \ \sum_{j=1}^{n} |x_j| < 1, |y| < \infty.$$

Knowing that the confluent function $\mathrm{H}_A^{(n,1)}$ for $y = 0$ is equal to the Lauricella function $F_A^{(n)}$, in the future it is required to prove and apply the theorem on the behavior of a confluent function $\mathrm{H}_A^{(n,1)}$.

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