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Boundary Value Problem for a Nonlinear Partial Differential Equation

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Abstract. *This paper investigates a boundary value problem for a nonlinear seventh-order partial differential equation with a nonlocal integral term. The problem is considered in a rectangular domain and is supplemented with Samarskii–Ionkin type boundary conditions with respect to the spatial variable and two-point boundary conditions with respect to time. Owing to the non-self-adjoint nature of the associated spatial operator, the corresponding spectral and adjoint spectral problems are studied in detail. Explicit representations of eigenvalues and eigenfunctions are obtained, and the biorthogonality, completeness, minimality, and Riesz basis properties of the resulting systems are established.*
By employing the Fourier method based on the biorthogonal systems of eigenfunctions, the original boundary value problem is reduced to a countable system of nonlinear integral equations. Sufficient conditions guaranteeing the existence and uniqueness of solutions are derived by means of the method of successive approximations and the contraction mapping principle. Furthermore, convergence of the constructed Fourier series and its derivatives is rigorously justified. As a consequence, a classical solution to the boundary value problem is obtained in an explicit series form. The results contribute to the theory of nonlocal higher-order partial differential equations with non-self-adjoint boundary conditions and provide an effective analytical framework for related nonlinear mathematical models.

Key words: *Boundary value problem, seventh order nonlinear equation, adjoint spectral problem, eigenfunctions, biorthogonal systems, unique solvability.*

MSC 2020: 35A01, 35A02, 35S15.

Introduction. Problem statement

Higher-order partial differential equations arise naturally in many branches of mathematical physics, continuum mechanics, filtration theory, hydrodynamics, elasticity, and engineering sciences. The mathematical analysis of such equations is often complicated by the presence of nonlocal terms, non-standard boundary conditions, and nonlinear interactions. These factors significantly influence both the qualitative behavior of solutions and the mathematical methods required for their investigation.

Particular interest has recently been devoted to boundary value problems involving non-self-adjoint operators and nonlocal conditions. Such problems frequently appear in applications where conservation laws, memory effects, spatial interactions, or integral constraints must be incorporated into the mathematical model. In these situations, classical orthogonal eigenfunction expansions are generally unavailable, and alternative approaches based on biorthogonal systems become necessary.

The present paper studies a nonlinear seventh-order partial differential equation containing an integral functional with respect to the spatial variable. The equation is supplemented by Samarskii–Ionkin type boundary conditions and two-point conditions in time. Problems of this type can be viewed as higher-order generalizations of mathematical models arising in filtration processes in fractured porous media and in related nonlocal evolution phenomena.

The principal difficulty of the problem lies in the non-self-adjoint character of the corresponding spectral operator. Therefore, a detailed spectral analysis is carried out. The associated adjoint problem is constructed, and the corresponding systems of eigenfunctions are investigated. It is shown that these systems form biorthogonal Riesz bases in the underlying functional space, which provides the foundation for the application of the Fourier method.

Using the method of separation of variables and biorthogonal Fourier expansions, the original boundary value problem is transformed into a countable system of nonlinear integral equations. The solvability of this system is studied by fixed-point techniques. Appropriate sufficient conditions are established to guarantee existence and uniqueness of solutions, while convergence properties of the resulting Fourier series are analyzed in detail.

The obtained results extend earlier investigations devoted to higher-order equations with classical boundary conditions and demonstrate the effectiveness of spectral methods in the study of nonlinear nonlocal equations with Samarskii–Ionkin type constraints. The developed approach may also be useful for a wider class of nonlocal evolution problems governed by higher-order differential operators.

However, differential equations of mathematical physics have a wide range of applications in the sciences and technology (see, for examples [1]–[8]). According to applications this field of mathematics is developing at a rapid pace and a lot of works are being published (see, for examples [9]–[15]).

The main equations of the theory of non-stationary filtration in fractured-pore formations are formulated in the work of G. I. Barenblatt, Yu. P. Zheltov and I. N. Kochina [16] (see, also [17]) and, further, developed by many authors [18]–[23]. Boussinesq type partial differential equations with boundary conditions are studied partially, in [24]–[27].

The spectral problems are considered in the works [28]–[31].

In the works [32]–[34], the similar equation is studied by Cauchy and Dirichlet conditions. So, our new work, which presents in this paper, is further development of the works [32]–[34].

In this paper, we study the classical solvability of the boundary value problem for a seventh order partial differential equation. In the rectangular domain, $\Omega = \{0 < t < T,\, 0 < x < 1\}$ we consider the following partial differential equation

$$\left[\frac{\partial^3}{\partial t^3}+\frac{\partial^7}{\partial t^3\partial x^4}+\omega^3\frac{\partial^4}{\partial x^4}\right]U(t,x)=f\left(t,x,\int\limits_0^1 q(y)U(t,y)dy\right), \qquad (0.1)$$

where T is given positive number, ω is nonzero real parameter.

In solving partial differential equation (0.1) we use Samarskii–Ionkin type boundary value conditions

$$U(t,1)=0,\quad U_{xx}(t,0)=0,\quad U_x(t,0)=U_x(t,1),\quad U_{xxx}(t,0)=U_{xxx}(t,1),\quad 0\le t\le T \qquad (0.2)$$

and two-point boundary value conditions

$$U(0,x)=\varphi_1(x),\quad U(T,x)=\varphi_2(x),\quad U'(0,x)=\varphi_3(x), \qquad (0.3)$$

where $\varphi_i(x)$ is enough smooth on the segment $[0,1]$.

Problem 0.1. To find a function $U(t,x)\in C(\overline{\Omega})\cap C^{3,4}_{t,x}(\Omega)$, which satisfies differential equation (0.1) and the conditions (0.2), (0.3), where $\overline{\Omega}=\{0\le t\le T,\ 0\le x\le 1\}$.

This problem is further development of the work [35].

1 Spectral Problems

We consider homogeneous differential equation

$$\frac{\partial U^3(t,x)}{\partial t^3}+\frac{\partial^7 U(t,x)}{\partial t^3\,\partial x^4}+\omega^3\frac{\partial^4 U(t,x)}{\partial x^4}=0. \qquad (1.1)$$

We will look for a non-trivial particular solution of the equation in the form $U(t,x)=u(t)\cdot\vartheta(x)$. Substituting this product of functions, depending from different variables, into equation (1.1), we obtain

$$-\frac{u'''(t)}{u'''(t)+\omega^3 u(t)}=\frac{\vartheta^{(IV)}(x)}{\vartheta(x)}.$$

Hence, equating second fraction into λ we obtain

$$\vartheta^{(IV)}(x)-\lambda\vartheta(x)=0,\quad \lambda\ge 0. \qquad (1.2)$$

Using conditions (0.2), from product of two functions we obtain conditions for the eigenvalues λ and eigenfunctions $\vartheta(x)$:

$$\vartheta(1)=0,\quad \vartheta''(0)=0,\quad \vartheta'(0)=\vartheta'(1),\quad \vartheta'''(0)=\vartheta'''(1). \qquad (1.3)$$

Solving the spectral problem (1.2), (1.3), we derive the eigenvalues [36]–[38]

$$\lambda_n=(2\pi n)^4,\quad n=0,1,2,... \qquad (1.4)$$

Eigenfunctions, corresponding to the eigenvalues (1.4), have the forms

$$\vartheta_0(x)=2(1-x),\quad \vartheta_{1n}(x)=-2\sin 2\pi nx,\quad \vartheta_{2n}(x)=\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx. \qquad (1.5)$$

The spectral problem (1.2), (1.3) is not self-adjoint and it is easy to see that the following problem with the eigenvalues (1.4) will be adjoint to it

$$\sigma^{(IV)}(x) - \lambda \sigma(x) = 0, \quad 0 < x < 1, \tag{1.6}$$

$$\sigma(0) = \sigma(1), \quad \sigma'(1) = 0, \quad \sigma''(0) = \sigma''(1), \quad \sigma'''(0) = 0. \tag{1.7}$$

We also consider adjoint problem (1.6), (1.7). Solving this problem, it is not difficult to see that the eigenfunctions, corresponding to eigenvalues (1.4), have the form

$$\sigma_0(x) = 1, \quad \sigma_{1n}(x) = \frac{e^{2\pi n x} + e^{2\pi n(1-x)}}{e^{2\pi n} - 1} - \sin 2\pi n x, \quad \sigma_{2n}(x) = -2\cos 2\pi n x. \tag{1.8}$$

Lemma 1.1 ([39])**.** *Systems of functions* (1.5) *and* (1.8) *are biorthogonal systems in* $L_2[0,1]$*:*

$$\left(\vartheta_0, \sigma_0\right)_0 = 1, \ \left(\vartheta_{ik}, \sigma_{jn}\right)_0 = \begin{cases} 1, & k = n, \ i = j \\ 0, & k \neq n, \ i \neq j \end{cases}, \quad i,j = 1,2, \quad n,k = 1,2,...$$

Lemma 1.2 ([39])**.** *The systems of functions* (1.5) *and* (1.8) *are minimal in* $L_2[0,1]$.

Lemma 1.3 ([39])**.** *The system of functions* (1.5) *and* (1.8) *is complete in the space* $L_2[0,1]$.

Lemma 1.4 ([39])**.** *The system of functions* (1.5) *and* (1.8) *forms the Riesz basis in* $L_2[0,1]$.

2 Formal solution of the problem (0.1)-(0.3)

Taking into account the formulas (1.5) and (1.8) we look for a solution to the problem (0.1)–(0.3) in the form of following Fourier series:

$$U(t,x) = u_0(t)\,\vartheta_0(x) + \sum_{n=1}^{\infty} \Big(u_{1,n}(t)\,\vartheta_{1,n}(x) + u_{2,n}(t)\,\vartheta_{2,n}(x) \Big), \tag{2.1}$$

where

$$u_0(t) = \int_0^1 U(t,y)\sigma_0(y)dy, \quad u_{\kappa,n}(t) = \int_0^1 U(t,y)\sigma_{\kappa,n}(y)dy, \quad \kappa = 1,2. \tag{2.2}$$

Similarly, the function $f\left(t,x,\int_0^1 q(y)U(t,y)dy\right)$ we write as

$$f\left(t,x,\int_0^1 q(y)U(t,y)dy\right) = f_0(t,u)\,\vartheta_0(x) + \sum_{n=1}^{\infty} \Big(f_{1,n}(t,u)\,\vartheta_{1,n}(x) + f_{2,n}(t,u)\,\vartheta_{2,n}(x) \Big), \tag{2.3}$$

where

$$f_0(t,u) = \int_0^1 f\left(t,y,\int_0^1 q(z)U(t,z)dz\right)\sigma_0(y)dy, \quad f_{\kappa,n}(t,u) = \int_0^1 f\left(t,y,\int_0^1 q(z)U(t,z)dz\right)\sigma_{\kappa,n}(y)dy, \quad \kappa = 1,2. \tag{2.4}$$

Let the function $U(t,x)$ in (2.1) be a solution to boundary problem (0.1)–(0.3). Then, substituting representations (2.1), (2.3) into equation (0.1) and taking (2.2), (2.4) into account, we obtain

$$u_0'''(t)\vartheta_0(x)+\sum_{n=1}^{\infty}\Big(u_{1,n}'''(t)\,\vartheta_{1,n}(x)+u_{2,n}'''(t)\,\vartheta_{2,n}(x)\Big)+$$

$$+\sum_{n=1}^{\infty}\lambda_n\Big(u_{1,n}'''(t)\,\vartheta_{1,n}(x)+u_{2,n}'''(t)\vartheta_{2,n}(x)\Big)+\sum_{n=1}^{\infty}\omega^3\lambda_n\Big(u_{1,n}(t)\,\vartheta_{1,n}(x)+u_{2,n}(t)\vartheta_{2,n}(x)\Big)=$$

$$=f_0(t,u)\vartheta_0(x)+\sum_{n=1}^{\infty}\Big(f_{1,n}(t,u)\,\vartheta_{2,n}(x)+f_{2,n}(t,u)\,\vartheta_{2,n}(x)\Big).$$

Hence, we obtain two differential equations: scalar differential equation and a countable system of third order ordinary differential equations

$$u_0'''(t)=f_0\big(t,\int_0^1 q(x)u_0(t)\vartheta_0(x)\big), \tag{2.5}$$

$$u_{\kappa,n}'''(t)+\mu_n\omega^3u_{\kappa,n}(t)=\frac{f_{\kappa,n}\left(t,\int_0^1 q(x)\sum_{i=1}^{\infty}u_{\kappa,n}(t)\vartheta_{\kappa,n}(x)dx\right)}{1+\lambda_n}, \tag{2.6}$$

First, we solve the equation (2.5):

$$u_0(t)=C_{1,0}+tC_{2,0}+\frac{t^2}{2}C_{3,0}+\int_0^t\frac{(t-s)^2}{2}f_0(s,\int_0^1 q(x)u_0(s)\vartheta_0(x)dx)ds. \tag{2.7}$$

To find $C_{j,0}(j=1,2,3)$, we rewrite the conditions (0.3) as

$$u_0(0)=\varphi_{1,0},\quad u_0(T)=\varphi_{2,0},\quad u_0'(0)=\varphi_{3,0}, \tag{2.8}$$

where

$$\varphi_{j,0}=\int_0^1\varphi_j(y)\sigma_0(y)dy,\quad j=1,2,3.$$

By virtue of (2.8), from (2.7) we obtain that

$$C_{1,0}=\varphi_{1,0},\quad C_{2,0}=\varphi_{2,0},$$

$$C_{3,0}=\varphi_{3,0}-\frac{2}{T^2}\varphi_{1,0}-\frac{2}{T}\varphi_{2,0}-\frac{2}{T^2}\int_0^T\frac{(T-s)^2}{2}f_0(s,\int_0^1 q(x)u_0(s)\vartheta_0(x)dx)ds. \tag{2.9}$$

Substituting (2.9) into the representation (2.7), derived

$$u_0(t)=\varphi_{1,0}\left[1-\left(\frac{t}{T}\right)^2\right]+\varphi_{2,0}\left[t-\frac{t^2}{T}\right]+$$

$$+\left(\frac{t}{T}\right)^2\varphi_{3,0}+\int\limits_0^T K_0(t,s)f_0\big(s,\int\limits_0^1 q(x)u_0(s)\vartheta_0(x)dx\big)ds, \tag{2.10}$$

where

$$K_0(t,s)=\begin{cases}\dfrac{(t-s)^2}{2}-\dfrac{(T-s)^2}{T^2}, & 0\le s<t,\\ -\dfrac{(T-s)^2}{T^2}, & t\le s\le T.\end{cases}$$

Theorem 2.1. *Let the conditions are fulfilled:*

1). $\left|\,f(t,\int\limits_0^1 q(x)u_0(t)\vartheta_0(x)dx)\,\right|\le M_0<\infty;$

2). $\left|\,f\big(t,\int\limits_0^1 q(x)u_{01}(t)\vartheta_0(x)dx\big)-f_0\big(t,\int\limits_0^1 q(x)u_{02}(t)\vartheta_0(x)dx\big)\,\right|\le L_0\int\limits_0^1 q(x)\big|\,u_{01}(t)-u_{02}(t)\,\big|\cdot\big|\,\vartheta_0(x)\,\big|dx,$
$0<L_0=\text{const};$

3). $\rho=2L_0\max\limits_{0\le t\le T}\int\limits_0^T\big|\,K_0(t,s)\,\big|ds<1.$

Then the equation (2.10) *has a unique solution.*

The proof of the theorem 2.1 is easy to see it. So, we do not show the proof of this theorem.

We will integrate the countable system of third-order differential equations (2.6) (see, [**?**]).

The solution that satisfies the equation (2.6) with the boundary conditions (2.8) we find as follows. The value of the parameter ω, for which the equation

$$\Delta_n=-\sqrt{3}\tau_n(\omega)\,\left(e^{-2\tau_n(\omega)T}+2e^{\tau_n(\omega)T}\sin\left(\sqrt{3}\tau_n(\omega)T-\frac{\pi}{6}\right)\right)=0,\quad \tau_n(\omega)=\frac{\sqrt[3]{\mu_n}\omega}{2}.$$

will have no solutions and $|\,\Delta_n\,|\ge\varepsilon_0>0$, we call as regular values of the parameter ω, and we will denote its set of solutions by Λ_0 and other values of parameter ω, we call as irregular values of the parameter: $\omega\in R\setminus(\Lambda_0\cup\{0\})$.

For regular values $\omega\in\Lambda_0$ we will continue to solve the countable system (2.6) and obtain:

$$u_{\kappa,n}(t,\omega)=\chi_{1,n}(t,\omega)\,\varphi_{\kappa,1,n}(\omega)+\chi_{2,n}(t,\omega)\,\varphi_{\kappa,2,n}(\omega)+\chi_{3,n}(t,\omega)\,\varphi_{\kappa,3,n}(\omega)+$$

$$+\frac{1}{1+\lambda_n}\int\limits_0^t K_{1,n}(t,s,\omega)\,f_{\kappa,n}(s,\omega,u)ds+\frac{1}{1+\lambda_n}\int\limits_0^T \bar{K}_{1,n}(T,s,\omega)\,f_{\kappa,n}\big(s,\omega,u\big)ds, \tag{2.11}$$

where

$$\chi_{1,n}(t,\omega)=\frac{2e^{\tau_n(\omega)(T+t)}\sin\left(\sqrt{3}\tau_n(\omega)(T-t)\right)}{\sqrt{3}\chi_{0,n}(T,\omega)},$$

$$\chi_{2,n}(t,\omega)=\frac{e^{-2\tau_n(\omega)t}-2e^{\tau_n(\omega)t}\cos\left(\sqrt{3}\tau_n(\omega)t+\dfrac{\pi}{6}\right)}{\chi_{0,n}(T,\omega)},\quad \chi_{3,n}(t,\omega)=$$

$$\frac{e^{\tau_n(\omega)(T-2t)}\sin\left(\sqrt{3}\tau_n(\omega)T\right)+e^{\tau_n(\omega)(t-2T)}\sin\left(\sqrt{3}\tau_n(\omega)t\right)+e^{\tau_n(\omega)(T+t)}\sin\left(\sqrt{3}\tau_n(\omega)(T-t)\right)}{\sqrt{3}\tau_n(\omega)\,\chi_{0,n}(T,\omega)},$$

$$K_{1,n}(t,s,\omega)=\frac{1}{12\tau_n^2(\omega)\chi_{0,n}(T,\omega)}e^{\tau_n(\omega)(t-s)}\left(e^{\tau_n(\omega)(t-s)}-2\cos\left(\sqrt{3}\tau_n(\omega)(t-s)+\frac{\pi}{3}\right)\right),$$

$$\bar{K}_{1,n}(T,s,\omega)=-\frac{\chi_{0,n}(t,\omega)}{12\tau_n^2(\omega)\chi_{0,n}(T,\omega)}\left(e^{2\tau_n(\omega)(T-s)}-2e^{\tau_n(\omega)(T-s)}\cos\left(\sqrt{3}\tau_n(\omega)(T-s)+\frac{\pi}{3}\right)\right).$$

The equation (2.11) we rewrite as

$$u_{\kappa,n}(t,\omega)=\chi_{1,n}(t,\omega)\,\varphi_{\kappa,1,n}(\omega)+\chi_{2,n}(t,\omega)\,\varphi_{\kappa,2,n}(\omega)+\chi_{3,n}(t,\omega)\,\varphi_{\kappa,3,n}(\omega)+$$

$$+\frac{1}{1+\lambda_n}\int_0^T K_{2,n}(t,s,\omega)\,f_{\kappa,n}\left(s,\omega,\int_0^1 q(x)\sum_{i=1}^{\infty}u_{\kappa,n}(s)\vartheta_{\kappa,n}(x)dx\right)ds, \tag{2.12}$$

where

$$K_{2,n}(t,s,\omega)=\begin{cases} K_{1,n}(t,s,\omega), & t\le s\le T,\\ K_{1,n}(t,s,\omega)+\bar{K}_{1,n}(t,s,\omega), & 0\le s<t.\end{cases}$$

Taking (2.12) into account, we have the following series

$$U(t,x,\omega)=2(1-x)\times$$

$$\times\left[\varphi_{1,0}\left[1-\left(\frac{t}{T}\right)^2\right]+\varphi_{2,0}\left[t-\frac{t^2}{T}\right]+\left(\frac{t}{T}\right)^2\varphi_{3,0}+\int_0^T K_0(t,s)f_0\left(s,\int_0^1 q((x)\omega,u_0(s)\vartheta_0(x)dx\right)ds\right]+$$

$$+\sum_{n=1}^{\infty}\sum_{\kappa=1}^{2}\Bigg[\chi_{1,n}(t,\omega)\,\varphi_{\kappa,1,n}(\omega)+\chi_{2,n}(t,\omega)\,\varphi_{\kappa,2,n}(\omega)+\chi_{3,n}(t,\omega)\,\varphi_{\kappa,3,n}(\omega)+$$

$$+\frac{1}{1+\lambda_n}\int_0^T K_{2,n}(t,s,\omega)\,f_{\kappa,n}\left(s,\omega,\int_0^1 q(y)\sum_{i=1}^{\infty}u_{\kappa,n}(t)\vartheta_{\kappa,n}(y)dy\right)ds\Bigg]\vartheta_{\kappa,n}(x),\quad \kappa=1,2. \tag{2.13}$$

3 Unique solvability of the countable system of integral equations (2.12)

Smoothness condition. Let in the domain $[0,1]$ the functions $\varphi_j(x)$ $(j=1,2,3)$ and $f(t,x)$ satisfied the conditions

$$\varphi_j(x)\in C_x^5[0,1],\quad \varphi_j(1)=\frac{d^2}{dx^2}\varphi_j(0)=\frac{d^4}{dx^4}\varphi_j(1)=0,$$

$$\frac{d}{dx}\varphi_j(0)=\frac{d}{dx}\varphi_j(1),\quad \frac{d^3}{dx^3}\varphi_j(0)=\frac{d^3}{dx^3}\varphi_j(1),$$

$$f(t,x,u)\in C_{t,x,u}^{0,2,0}([0,T]\times[0,1]\times R),\quad f(t,1,u)=\frac{d^2}{dx^2}(t,0,u)=\frac{d^4}{dx^4}f(t,1)=0,$$

$$\frac{d}{dx}f(t,0,u)=\frac{d}{dx}f(t,1,u),\quad \frac{d^3}{dx^3}f(t,0,u)=\frac{d^3}{dx^3}f(t,1,u).$$

Then, we integrate by parts

$$\varphi_{\kappa,j,n}=\int_0^1\varphi_j(y)\sigma_\kappa(y)dy,\quad \kappa=1,2,\quad j=1,2,3$$

five times on the variable y, respectively, and obtain

$$\varphi_{1,j,n} = -\left(\frac{1}{2\pi}\right)^5 \frac{\varphi_{1,j,n}^{(V)}}{n^5}, \quad j = 1,2,3,$$

$$\varphi_{1,j,n}^{(V)} = \int_0^1 \frac{\partial^5 \varphi_j(y)}{\partial y^5} \left(\frac{e^{2\pi n y} - e^{2\pi n(1-y)}}{e^{2\pi n} - 1} + \cos 2\pi n y\right) dy,$$

$$\varphi_{2,j,n} = \left(\frac{1}{2\pi}\right)^5 \frac{\varphi_{2,j,n}^{(V)}}{n^5}, \quad \varphi_{2,j,n}^{(V)} = 2\int_0^1 \frac{\partial^5 \varphi_j(y)}{\partial y^5} \sin 2\pi n y dy.$$

Similarly, we have

$$f_{1,n}(t,u) = -\left(\frac{1}{2\pi}\right)^2 \frac{f_{1,n}''(t,u)}{n^2}, \quad f_{1,n}''(t,u) = \int_0^1 \frac{\partial^2 f(t,y,U)}{\partial y^2} \left(\frac{e^{2\pi n y} - e^{2\pi n(1-y)}}{e^{2\pi n} - 1} + \cos 2\pi n y\right) dy,$$

$$f_{2,n}(t,u) = -\left(\frac{1}{2\pi}\right)^2 \frac{f_{2,n}''(t,u)}{n^2}, \quad f_{2,n}''(t,u) = \int_0^1 \frac{\partial^2 f(t,y,U)}{\partial y^2} \sin 2\pi n y dy.$$

In addition, we say that the following estimates are true

$$\left\| \vec{\varphi}_{\kappa,j}^{(V)} \right\|_{\ell_2} \le N_1 \left\| \frac{\partial^5 \varphi_j(x)}{\partial x^5} \right\|_{L_2[0,1]},$$

$$\left\| \max_{0\le t\le T} \vec{f}_\kappa''(t,u) \right\|_{B_2} \le N_2 \left\| \max_{0\le t\le T} \frac{\partial^2 f(t,x,U)}{\partial x^2} \right\|_{L_2[0,1]}, \quad 0 < N_\kappa = \text{const}, \quad \kappa = 1,2.$$

Theorem 3.1. *Let the conditions are fulfilled:*

1). $\left| f_{\kappa,n}(t, u_{\kappa,n}(t)) \right| \le M_\kappa < \infty, \ \ \kappa = 1,2;$

2). $\left| f_{\kappa,n}\big(t, u_{\kappa,n,1}(t)\big) - f_{\kappa,n}\big(t, u_{\kappa,n,2}(t)\big) \right| \le L_\kappa(x) \left| u_{\kappa,n,1}(t) - u_{\kappa,n,2}(t) \right|, \ \ 0 < L_\kappa(x) \in L[0,1];$

3). $\rho = C_1(\omega)\tilde{C}_3 \left\| q(x) \right\|_{L_2[0,1]} \int_0^1 L_\kappa(y)\sigma_\kappa(y)dy < 1.$

Then the countable system of integral equations (2.12) *has a unique solution.*

Proof. We estimate the successive approximations for (2.12):

$$\left\| u_\kappa^{m+1}(t,\omega) \right\|_{B_2[0,T]} \le$$

$$\le \sum_{n=1}^\infty \max_{0\le t\le T} \Big[|\chi_{1,n}(t,\omega)| \, |\varphi_{\kappa,1,n}(\omega)| + |\chi_{2,n}(t,\omega)| \, |\varphi_{\kappa,2,n}(\omega)| + |\chi_{3,n}(t,\omega)| \, |\varphi_{\kappa,3,n}(\omega)| \Big] +$$

$$+ \sum_{n=1}^\infty \max_{0\le t\le T} \frac{1}{\lambda_n} \int_0^T |K_{2,n}(t,s,\omega)| \left| f_{\kappa,n}\left(s, \omega, \int_0^1 q(x) \sum_{i=1}^\infty u_{\kappa,i}^m(s)\vartheta_{\kappa,i}(x)dx\right) ds \right| \le$$

$$\leq C_0(\omega)\Big(\frac{1}{2\pi}\Big)^5\sum_{n=1}^{\infty}\Bigg[\frac{\Big|\varphi^{(V)}_{\kappa,1,n}(\omega)\Big|}{n^5}+\frac{\Big|\varphi^{(V)}_{\kappa,2,n}(\omega)\Big|}{n^5}+\frac{\Big|\varphi^{(V)}_{\kappa,3,n}(\omega)\Big|}{n^5}\Bigg]+$$

$$+C_1(\omega)(2\pi)^2\sum_{n=1}^{\infty}\max_{0\leq t\leq T}\frac{\Big|f''_{\kappa,n}\Big(t,\omega,\int\limits_0^1 q(x)\sum\limits_{i=1}^{\infty}u^m_{\kappa,i}(t)\vartheta_{\kappa,i}(x)dx\Big)\Big|}{n^6}\leq$$

$$\leq C_0(\omega)C_2\Bigg[\sqrt{\sum_{n=1}^{\infty}\Big|\varphi^{(V)}_{\kappa,1,n}(\omega)\Big|^2}+\sqrt{\sum_{n=1}^{\infty}\Big|\varphi^{(V)}_{\kappa,2,n}(\omega)\Big|^2}+\sqrt{\sum_{n=1}^{\infty}\Big|\varphi^{(V)}_{\kappa,3,n}(\omega)\Big|^2}\Bigg]+$$

$$+C_1(\omega)C_3\sqrt{\sum_{n=1}^{\infty}\max_{0\leq t\leq T}\Bigg|f''_{\kappa,n}\Bigg(t,\omega,\int\limits_0^1 q(x)\sum_{i=1}^{\infty}u^m_{\kappa,i}(t)\vartheta_{\kappa,i}(x)dx\Bigg)\Bigg|^2}\leq$$

$$\leq C_0(\omega)C_2\Bigg[\Big\|\varphi^{(V)}_{\kappa,1}(\omega)\Big\|_{\ell_2}+\Big\|\varphi^{(V)}_{\kappa,2}(\omega)\Big\|_{\ell_2}+\Big\|\varphi^{(V)}_{\kappa,3}(\omega)\Big\|_{\ell_2}\Bigg]+$$

$$+C_1(\omega)C_3\max_{0\leq t\leq T}\Bigg\|f''_{\kappa}\Bigg(t,\omega,x,\int\limits_0^1 q(y)\sum_{i=1}^{\infty}u^m_{\kappa,i}(t)\vartheta_{\kappa,i}(y)dy\Bigg)\Bigg\|_{L_2(\overline{\Omega})}<\infty, \tag{3.1}$$

where
$C_0(\omega)=\max\Big\{\max\limits_{0\leq t\leq T}\max\limits_{n=1,2,\dots}|\chi_{1,n}(t,\omega)|+\max\limits_{0\leq t\leq T}\max\limits_{n=1,2,\dots}|\chi_{2,n}(t,\omega)|+\max\limits_{0\leq t\leq T}\max\limits_{n=1,2,\dots}|\chi_{1,n}(t,\omega)|\Big\}$,
$C_1(\omega)=\max\limits_{0\leq t\leq T}\max\limits_{n=1,2,\dots}\int\limits_0^T|K_{2,n}(t,s,\omega)|ds$, $C_2=\Big(\frac{1}{2\pi}\Big)^5\sqrt{\sum\limits_{n=1}^{\infty}\frac{1}{n^{10}}}$, $C_3=\big(2\pi\big)^2\sqrt{\sum\limits_{n=1}^{\infty}\frac{1}{n^{12}}}$.

For the difference of the two successive approximations we obtain

$$||\,\vec{u}^{m+1}_{\kappa}(t,\omega)-\vec{u}^{m}_{\kappa}(t,\omega)\,||_{B_2[0,T]}\leq C_1(\omega)\tilde{C}_3\times$$

$$\times\max_{0\leq t\leq T}\Bigg\|f_{\kappa}\Bigg(t,\omega,x,\int\limits_0^1 q(y)\sum_{i=1}^{\infty}u^m_{\kappa,i}(t)\vartheta_{\kappa,i}(y)dy\Bigg)-f_{\kappa}\Bigg(t,\omega,x,\int\limits_0^1 q(y)\sum_{i=1}^{\infty}u^{m-1}_{\kappa,i}(t)\vartheta_{\kappa,i}(y)dy\Bigg)\Bigg\|_{L_2}\leq$$

$$\leq C_1(\omega)\tilde{C}_3\max_{0\leq t\leq T}\int\limits_0^1\Bigg|f\Bigg(t,\omega,y,\int\limits_0^1 q(z)\sum_{i=1}^{\infty}u^m_{\kappa,i}(t)\vartheta_{\kappa,i}(z)dz\Bigg)-$$

$$-f\Bigg(t,\omega,y,\int\limits_0^1 q(z)\sum_{i=1}^{\infty}u^{m-1}_{\kappa,i}(t)\vartheta_{\kappa,i}(z)dz\Bigg)\Bigg|\sigma_{\kappa}(y)dy\leq$$

$$\leq C_1(\omega)\tilde{C}_3\int\limits_0^1 L_{\kappa}(y)\sigma_{\kappa}(y)dy\int\limits_0^1 q(z)\sum_{i=1}^{\infty}\Big|u^{m-1}_{\kappa,i}(t)\vartheta_{\kappa,i}(z)-u^{m-1}_{\kappa,i}(t)\vartheta_{\kappa,i}(z)\Big|dz\leq$$

$$\leq C_4\sum_{i=1}^{\infty}\Big|u^{m-1}_{\kappa,i}(t)-u^{m-1}_{\kappa,i}(t)\Big|\Bigg|\int\limits_0^1 q(z)\vartheta_{\kappa,i}(z)dz\Bigg|\leq$$

$$\leq \rho \Big\| \vec{u}_\kappa^{m-1}(t) - \vec{u}_\kappa^{m-1}(t) \Big\|_{B_2[0,T]}, \tag{3.2}$$

where

$$\rho = C_1(\omega)\tilde{C}_3 \| q(x) \|_{L_2[0,1]} \int_0^1 L_\kappa(y)\sigma_\kappa(y)dy, \quad \tilde{C}_3 = \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^8}}.$$

From the estimates (3.1) and (3.2) implies that the Theorem 3.1 is true. □

4 Uniqueness of the solution to problem (0.1)-(0.3)

Theorem 4.1. *If there exists a solution of Problem* (0.1)–(0.3)*, then it is unique for regular values of the parameter* ω *from the set* Λ_0.

Proof. We consider the regular values of the parameter ω from the set Λ_0. Suppose that there exist two different solutions $U_1(t,x)$ and $U_2(t,x)$ to the problem (0.1)–(0.3). Then the difference $U(t,x) = U_1(t,x) - U_2(t,x)$ is a solution of the equation (0.1), satisfying the conditions (0.2) and (0.3) with functions $\varphi(x) \equiv 0$, $f(t,x) \equiv 0$. Then, it follows from formulas (2.7)–(2.9) and (2.13) that

$$u_{\kappa,n}(t) = \int_0^1 U(t,y)\sigma_{\kappa,n}(y)dy \equiv 0, \quad \kappa = 1, 2.$$

From this, due to the completeness of the system (1.5) and (1.8) in the space $L_2[0,1]$, it follows that $U(t,x) = 0$ almost everywhere on $[0,1]$ for any $t \in [0,T]$. Since $U(t,x) \in C(\bar{\Omega})$, it follows that $U(x,t) \equiv 0$ in $\bar{\Omega}$. The Theorem 4.1 is proved. □

5 Convergence of the series (2.13)

Theorem 5.1. *Let the smoothness conditions be satisfied. Then, for the regular values of the parameter* ω *from the set* Λ_0 *the function* (2.13) *will be belonged to the class* $U(t,x) \in C(\overline{\Omega}) \cap C^{3,4}_{t,x}(\Omega)$.

Proof. We consider the series (2.13) and

$$\frac{\partial^3}{\partial t^3} U(t,x,\omega) =$$

$$+ \sum_{n=1}^{\infty}\sum_{\kappa=1}^{2} \Big[\chi'''_{1,n}(t,\omega)\,\varphi_{\kappa,1,n}(\omega) + \chi'''_{2,n}(t,\omega)\,\varphi_{\kappa,2,n}(\omega) + \chi'''_{3,n}(t,\omega)\,\varphi_{\kappa,3,n}(\omega) +$$

$$+ \frac{1}{1+\lambda_n} \int_0^T K'''_{2,n}(t,s,\omega) f_{\kappa,n}(s)ds \Big] \vartheta_{\kappa,n}(x), \tag{5.1}$$

$$\frac{\partial^4}{\partial x^4} U(t,x,\omega) =$$

$$+ \sum_{n=1}^{\infty}\sum_{\kappa=1}^{2} \Big[\chi_{1,n}(t,\omega)\,\varphi_{\kappa,1,n}(\omega) + \chi_{2,n}(t,\omega)\,\varphi_{\kappa,2,n}(\omega) + \chi_{3,n}(t,\omega)\,\varphi_{\kappa,3,n}(\omega) +$$

$$+\frac{1}{1+\lambda_n}\int_0^T K_{2,n}(t,s,\omega)f_{\kappa,n}(s)ds\Bigg]\vartheta_{\kappa,n}^{(IV)}(x), \tag{5.2}$$

where

$$\vartheta_{1,n}^{(IV)}(x)=\big(-2\sin 2\pi nx\big)^{(IV)}=\big(2\pi\big)^4 n^4\vartheta_{1,n}(x),$$

$$\vartheta_{2,n}^{(IV)}(x)=\left(\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx\right)^{(IV)}=\big(2\pi\big)^4 n^4\vartheta_{2,n}(x).$$

The proof of convergence of the series (2.13) and (5.1) are implies from the estimates (3.1) and (3.2). So, we show the convergence of the series (5.2).

To proof of convergence of the series (5.2), we will prove the convergence for the following series

$$\sum_{n=1}^{\infty} n^4\Bigg[\chi_{1,n}'''(t,\omega)\,\varphi_{1,1,n}(\omega)+\chi_{2,n}'''(t,\omega)\,\varphi_{1,2,n}(\omega)+\chi_{3,n}'''(t,\omega)\,\varphi_{1,3,n}(\omega)+$$

$$+\frac{1}{1+n^4}\int_0^T K_{2,n}'''(t,s,\omega)f_{1,n}(s)ds\Bigg]\sin 2\pi nx, \tag{5.3}$$

$$\sum_{n=1}^{\infty} n^4\Bigg[\chi_{1,n}'''(t,\omega)\,\varphi_{2,1,n}(\omega)+\chi_{2,n}'''(t,\omega)\,\varphi_{2,2,n}(\omega)+\chi_{3,n}'''(t,\omega)\,\varphi_{2,3,n}(\omega)+$$

$$+\frac{1}{1+n^4}\int_0^T K_{2,n}'''(t,s,\omega)f_{2,n}(s)ds\Bigg]\left(\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx\right). \tag{5.4}$$

Applying the smoothness conditions and Bessel inequalities to (5.3), we have:

$$\max_{n=1,2,\dots}\big|\,\chi_{1,n}'''(t,\omega)\,\big|\sum_{n=1}^{\infty} n^4\,|\,\varphi_{1,1,n}(\omega)\,|+\max_{n=1,2,\dots}\big|\,\chi_{2,n}'''(t,\omega)\,\big|\sum_{n=1}^{\infty} n^4\,|\,\varphi_{1,2,n}(\omega)\,|+$$

$$+\max_{n=1,2,\dots}\big|\,\chi_{3,n}'''(t,\omega)\,\big|\sum_{n=1}^{\infty} n^4\,|\,\varphi_{1,3,n}(\omega)\,|+\sum_{n=1}^{\infty} n^4\frac{1}{1+n^4}\int_0^T\big|\,K_{2,n}'''(t,s,\omega)\,\big|\,|\,f_{1,n}(s,u)\,|\,ds\le$$

$$\le\max_{n=1,2,\dots}\max_{0\le t\le T}\big|\,\chi_{1,n}'''(t,\omega)\,\big|\sum_{n=1}^{\infty}\left|\frac{\varphi_{1,1,n}^{(V)}(\omega)}{n}\right|+\max_{n=1,2,\dots}\max_{0\le t\le T}\big|\,\chi_{2,n}'''(t,\omega)\,\big|\sum_{n=1}^{\infty}\left|\frac{\varphi_{1,2,n}^{(V)}(\omega)}{n}\right|+$$

$$+\max_{n=1,2,\dots}\max_{0\le t\le T}\big|\,\chi_{3,n}'''(t,\omega)\,\big|\sum_{n=1}^{\infty}\left|\frac{\varphi_{1,3,n}^{(V)}(\omega)}{n}\right|+\max_{n=1,2,\dots}\max_{0\le t\le T}\int_0^T\big|\,K_{2,n}'''(t,s,\omega)\,\big|\,ds\sum_{n=1}^{\infty}|\,f_{1,n}(t,u)\,|\le$$

$$\le\sum_{j=1}^{3}\delta_j(\omega)\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}\sqrt{\sum_{n=1}^{\infty}\left|\,\varphi_{1,j,n}^{(V)}(\omega)\right|^2}+\delta_K(\omega)\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^4}}\sqrt{\sum_{n=1}^{\infty}\max_{0\le t\le T}\left|\,f_{1,n}''(t,u)\right|^2}\le$$

$$\le\frac{\pi^2}{6}\left[N_1\sum_{j=1}^{3}\delta_j(\omega)\left\|\frac{\partial^5\varphi_j(x)}{\partial x^5}\right\|_{L_2[0,1]}+N_2\delta_K(\omega)\max_{0\le t\le T}\left\|\frac{\partial^2 f\big(t,x,U(t,x)\big)}{\partial x^2}\right\|_{L_2[0,1]}\right]<\infty,$$

where

$$\delta_j(\omega) = \max_{n=1,2,\dots} \max_{0\le t\le T} \left| \chi'''_{j,n}(t,\omega) \right|, \quad j=1,2,3; \qquad \delta_K(\omega) = \max_{n=1,2,\dots} \max_{0\le t\le T} \int\limits_0^T \left| K'''_{2,n}(t,s,\omega) \right| ds.$$

The prove that the second series (5.4) converges is similarly to the proof of the series (5.3). The theorem 5.1 is proved. □

Conclusion

A boundary value problem for a nonlinear seventh-order partial differential equation with a nonlocal integral term has been investigated. The problem was considered under Samarskii–Ionkin type boundary conditions in the spatial variable and two-point boundary conditions in time. Due to the non-self-adjoint nature of the underlying differential operator, special attention was devoted to the corresponding spectral analysis.

The associated spectral and adjoint spectral problems were solved explicitly. The eigenvalues and eigenfunctions were constructed, and the resulting systems were shown to possess biorthogonality, completeness, minimality, and Riesz basis properties. These results made it possible to apply the Fourier method in a non-self-adjoint setting and to represent the solution as a biorthogonal Fourier series.

The original nonlinear boundary value problem was reduced to a countable system of nonlinear integral equations. By employing the method of successive approximations and the contraction mapping principle, sufficient conditions ensuring existence and uniqueness of solutions were established. In addition, convergence of the Fourier expansions and their derivatives was rigorously justified, which guaranteed the classical solvability of the problem.

The theoretical results obtained in this work enrich the spectral theory of non-self-adjoint boundary value problems and contribute to the analysis of higher-order nonlinear partial differential equations with nonlocal effects. The proposed methodology can be extended to broader classes of evolution equations, including equations with variable coefficients, multidimensional domains, nonlinear boundary conditions, and more general integral operators.

Future investigations may focus on stability analysis, asymptotic behavior of solutions, inverse problems, numerical implementations, and applications to mathematical models arising in porous media, fluid mechanics, elasticity theory, and other areas of applied science.

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