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Optimal Control for a Nonlinear System of Integro-Differential Equations of Fractional Order

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Abstract. *This paper investigates an optimal control problem for a nonlinear system of fractional-order integro-differential equations involving the Gerasimov–Caputo operator. The considered model contains a nonlinear integral interaction represented by the product of two nonlinear functions, which substantially complicates both the analytical structure of the system and the derivation of optimality conditions.*
Unlike the classical approach in optimal control theory, where the control function is assumed to be known when determining the state trajectory, the present work constructs the state function and the control function simultaneously as a coupled solution pair of a nonlinear functional-integral system.
Using fractional integration techniques, the original fractional integro-differential equation is reduced to an equivalent nonlinear integral equation for the state function. By introducing an adjoint final-value problem and constructing the Pontryagin function, a second nonlinear integral equation for the control function is derived. As a result, the optimal control problem is transformed into a strongly coupled nonlinear system of integral equations.
A constructive analytical method based on successive approximations and the contraction mapping principle is developed. Explicit sufficient conditions ensuring existence and uniqueness of the vector-valued control and state functions are established. Furthermore, continuous dependence of the solution on the initial vector is proved.
The obtained results provide a constructive framework for the analysis of nonlinear fractional optimal control systems with memory effects and nonlocal interactions and extend several known solvability results for fractional dynamical systems.

Key words: *Fractional equation, product of a nonlinear function with a nonlinear integral, Pontryagin function, method of contracted mappings, existence, uniqueness and stability of a solution.*

1 Introduction. Problem statement

In the theory of differential equations, various methods for their solution have been developed [1]–[19]. In recent years, substantial attention has been devoted to optimal control problems governed by integer and fractional-order systems (see, for example, [20]–[37]). Fractional models often provide a more realistic description of complex dynamical processes than classical integer-order equations because the fractional operators incorporate memory effects and long-range temporal interactions directly into the mathematical model.

At the same time, nonlinear optimal control problems for fractional integro-differential equations remain analytically challenging, especially when the system contains nonlinear integral interactions and products of nonlinear functions. Such structures appear naturally in distributed parameter systems, nonlinear feedback processes, hereditary dynamics, and nonlocal control models.

Most existing works in fractional optimal control theory determine the state trajectory for a fixed control function and subsequently optimize the control parameter. In contrast, the present paper develops a different approach in which the state function and the control function are constructed simultaneously as a coupled solution pair of a nonlinear functional system. This leads to a substantially more complicated analytical structure because each equation depends on both unknown quantities.

Fractional differential and integro-differential equations have become one of the principal mathematical tools for describing systems with memory, hereditary effects, anomalous diffusion, and nonlocal interactions. Such equations naturally arise in viscoelasticity, continuum mechanics, biological systems, thermal processes, control theory, signal analysis, and many other areas of modern applied science (see, for example, [38]–[66]). The considered model involves the Gerasimov–Caputo fractional operator, which is especially important in applications due to its physically meaningful formulation of initial conditions. The nonlinear system also contains a Volterra-type integral interaction represented by the product of two nonlinear functions, which further increases the analytical complexity of the problem.

Motivated by these considerations, in this paper we investigate an optimal control problem for a nonlinear fractional-order integro-differential system. The original problem is reduced to a coupled nonlinear system consisting of two integral equations: one equation determines the state function, while the second equation determines the optimal control function through the Pontryagin functional framework.

The main contributions of the paper are the following:

- reduction of the original fractional optimal control problem to a coupled nonlinear functional-integral system;
- derivation of an integral equation for the control function using the Pontryagin function and an adjoint final-value problem;
- development of a constructive iterative procedure based on successive approximations;
- proof of existence and uniqueness of the coupled state-control solution pair by means of the contraction mapping principle;
- establishment of continuous dependence of the state function on the initial vector.

The developed approach provides an effective analytical framework for studying nonlinear fractional optimal control systems with memory effects, nonlinear integral interactions, and nonlocal dynamics.

On a given interval $[0, T]$, we consider the optimal control problem for the following system of nonlinear integro-differential equations of fractional order:

$$ {}_C D_{0t}^{\alpha} x(t) = g(t, x(t)) \int\limits_0^t f\left(s, x(s), u(s)\right) ds \tag{1.1} $$

with initial condition

$$ x(0) = A, \tag{1.2} $$

where $x(t)$ is state function, $u(t)$ is control function, by ${}_C D_{0t}^{\alpha}$ is denoted Gerasimov–Caputo fractional operator

$$ {}_C D_{0t}^{\alpha} x(t) = I_{0t}^{1-\alpha} x'(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{x'(s) ds}{(t-s)^{\alpha}}, \ \ t \in (0, T), $$

and

$$ I_{0t}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} x(s) ds, \quad 0 < \alpha \leq 1 $$

is Riemann–Liouville integral operator, α is order of fractional differential operator, $\Gamma(\alpha)$ is Gamma function: $\Gamma(z) = \int\limits_0^{\infty} t^{z-1} e^{-t} dt$, $\operatorname{Re} z > 0$, $A \in \mathbb{R}^n$ is a given non-zero vector function, $g \in C\big([0, T] \times X, \mathbb{R}^n\big)$, $f \in C\big([0, T] \times X \times \Upsilon, \mathbb{R}^n\big)$, X, Υ are closed sets in $\mathbb{R}^n$.

By $C\left([0, T], \mathbb{R}^n\right)$ is denoted Banach space, which consists continuous vector functions $x(t)$, defined on the segment $[0, T]$, with values in $\mathbb{R}^n$ and with the norm

$$ \| \, x \, \|_{C[0,T]} = \sqrt{\sum_{j=1}^{n} \max_{0 \leq t \leq T} | \, x_j(t) \, |^2}. $$

We consider the function

$$ u(t) \in \Upsilon = \{ u : \, | \, u(t) \, | \leq M*, \ t \in [0, T], \ 0 < M* = \text{const} \} $$

and the following functional of quality

$$ J[u] = [A - \beta]^2 + \gamma \int\limits_0^T \varepsilon(t) u^2(t) dt, \tag{1.3} $$

where $\beta \neq 0$, $0 < \gamma = \text{const}$, $0 < \varepsilon(t) \in C[0, T]$.

Problem 1.1. Find a control function $u^*(t) \in \Upsilon$ and corresponding values of state function $x^*(t) \in C\left([0, T], \mathbb{R}^n\right)$, providing a minimum of functionality (1.3) and for all $t \in [0, T]$, satisfies the given fractional integro-differential system (1.1), initial condition (1.2).

2 Reduction the initial problem to the integral equation. First relations between two unknown functions

Let the function $x(t) \in C\left([0,T],\mathbb{R}^n\right)$ be the solution to the problem (1.1)–(1.2). We act the operator I_{0t}^{α} to the equation (1.1)

$$I_{0t}^{\alpha}{}_{C}D_{0t}^{\alpha}x(t) = I_{0t}^{\alpha}g(t,x(t))\int\limits_{0}^{t} f\left(s,x(s),u(s)\right)ds$$

and use the definition of the Gerasimov–Caputo fractional operator

$$I_{0t}^{\alpha}{}_{C}D_{0t}^{\alpha}\left(x(t)\right) = I_{0t}^{\alpha}I_{0t}^{1-\alpha}x'(t) = I_{0t}^{1}x'(t) = \int\limits_{0}^{t} x'(s)ds = x(t) - x(0), \tag{2.1}$$

$$I_{0t}^{\alpha}\eta\left(t,x(t)\right) = \frac{1}{\Gamma(\alpha)}\int\limits_{0}^{t}(t-s)^{\alpha-1}\eta\left(s,x(s)\right)ds. \tag{2.2}$$

Then, applying the formulas (2.1) and (2.2) to the last fractional integro-differential equation, we obtain

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)}\int\limits_{0}^{t}(t-s)^{\alpha-1}g(s,x(s))\int\limits_{0}^{s} f\left(\theta,x(\theta),u(\theta)\right)d\theta ds.$$

Taking into account the condition (1.2), we will rewrite the last equality as follows

$$x(t) = K_1(t;x,u) \equiv A + \frac{1}{\Gamma(\alpha)}\int\limits_{0}^{t}(t-s)^{\alpha-1}g(s,x(s))\int\limits_{0}^{s} f\left(\theta,x(\theta),u(\theta)\right)d\theta ds \tag{2.3}$$

for all $t \in [0,T]$.

It is easy to verify that the equation (2.3) satisfies the problem (1.1)–(1.2). We differentiate the function (2.3) with respect to t:

$$x'(t) = \frac{1}{\Gamma(\alpha)}\int\limits_{0}^{t}(1-s)^{\alpha-1}g(s,x(s))\int\limits_{0}^{s} f\left(\theta,x(\theta),u(\theta)\right)d\theta ds \tag{2.4}$$

for all $t \in [0,T]$.

3 Integral equation for control function. Second relations between two unknown functions

Let $u^*(t) \in \Upsilon$ be optimal control function

$$\Delta\, J[u^*(t)] = J[u^*(t) + \Delta u^*(t)] - J[u^*(t)] \geq 0,$$

where $u^*(t) + \Delta u^*(t) \in C[0,T]$.

Let us consider the equation (2.3) and the problem of finding the control function

$$u^*(t) \in \Upsilon = \{u^* : |\, u^*(t) \,| \le M*, \;\; t \in [0,T], \;\; 0 < M* = \text{const}\}$$

and corresponding state function $x^*(t)$ that minimizes the functional (1.3).

In optimal control theory, the following approach is often used. We consider the following function, which is called as Pontryagin function to obtain the conditions of optimality:

$$H(u(t), x(t)) = \psi(t) \frac{1}{\Gamma(\alpha)} \int\limits_0^t (1-s)^{\alpha-1} g(s, x(s)) \times$$

$$\times \int\limits_0^s f\left(\theta, x(\theta), u(\theta)\right) d\theta ds - \gamma\varepsilon(t) u^2(t), \tag{3.1}$$

where the function $\psi(t)$ will be determined from the equation:

$$\psi'(t) = -\psi(t) \tag{3.2}$$

with final condition

$$\psi(T) = -2[A-\beta]. \tag{3.3}$$

Solving the differential equation (3.2), we obtain the function $\psi(t) = C_0 \exp\{-t\}$. In order to find the constant of integration C_0, we use the condition given at final point (3.3)

$$\psi(T) = C_0 \exp\{-T\} = -2[A-\beta].$$

Hence, taking into account the operator (2.3), we get the unknown coefficient

$$C_0 = -2[A-\beta] \exp\{T\}. \tag{3.4}$$

Consequently, by virtue of (3.4), we have

$$\psi(t) = -2[A-\beta] \exp\{T-t\}. \tag{3.5}$$

Substituting the values of function $\psi(t)$ from 12 into (3.1)

$$H(u(t), x(t)) = -\gamma\varepsilon(t) u^2(t) +$$

$$+ \frac{-2[A-\beta]\exp\{T-t\}}{\Gamma(\alpha)} \int\limits_0^t (1-s)^{\alpha-1} g(s, x(s)) \int\limits_0^s F\left(\theta, x(\theta), u(\theta)\right) d\theta ds. \tag{3.6}$$

By differentiation (3.6) with respect to control function from (3.6) we derive the integral equation with respect to control function

$$H_u(u(t), x(t)) = -2\gamma\varepsilon(t) u(t) +$$

$$+ \frac{-2[A-\beta]\exp\{T-t\}}{\Gamma(\alpha)} \int\limits_0^t (1-s)^{\alpha-1} g(s, x(s)) \int\limits_0^s f_u\left(\theta, x(\theta), u(\theta)\right) d\theta ds. \tag{3.7}$$

From the equation (3.7) we get a new equation for control function

$$u(t) = K_2(t; x, u) \equiv$$

$$\equiv a_0(t) \int_0^t (1-s)^{\alpha-1} g(s, x(s)) \int_0^s f_u\left(\theta, x(\theta), u(\theta)\right) d\theta ds, \tag{3.8}$$

where $a_0(t) = \dfrac{(\beta - A)\exp\{T - t\}}{\gamma\varepsilon(t)\Gamma(\alpha)}$.

4 Main results

We solve the integral equations (2.3) and (3.8) as a system.

Theorem 4.1. *Let us assume that the following conditions are met:*

1). $M_g = \max\limits_{0\le t\le T} |\, g\,(t,x)\,| < \infty$; $M_{0,f} = \max\limits_{0\le t\le T} |\, f\,(t,x,u)\,| < \infty$;

2). $M_{1,f} = \max\limits_{0\le t\le T} |\, f_u\,(t,x,u)\,| < \infty$;

3). *For all* $t \in [0,T]$, *and* $u \in \Upsilon$ *there exist functions* $0 < L_\kappa(t) \in C[0,T]$, $\kappa = 1,2,3$, *such that there is true the inequality*

$$|\, g(t,x_1) - g(t,x_2)\,| \le L_1(t)\,|\, x_1 - x_2\,|;$$

$$|\, f(t,x_1,u_1) - f(t,x_2,u_2)\,| \le L_2(t)\left(|\, x_1 - x_2\,| + |\, u_1 - u_2\,|\right);$$

$$|\, f_u(t,x_1,u_1) - f_u(t,x_2,u_2)\,| \le L_3(t)\ \left(|\, x_1 - x_2\,| + |\, u_1 - u_2\,|\right);$$

4). $\rho = \chi_{11} + \chi_{21} < 1$, *where* χ_{11}, χ_{21} *are defined below.*

Then the system of integral equations of the state function (2.3) *and the optimization function* (3.8) *has a unique pair of solutions* $\left\{x(t) \in C\left([0,T], \mathbb{R}^n\right),\ u(t) \in \Upsilon\right\}$. *This pair of solutions can be found from the following iterative processes:*

$$\begin{cases} x^k\,(t) = K_1(t; x^{k-1}, u^{k-1}), \quad k = 1,2,3,... \\ x^0\,(t) = A, \quad t \in [0,T]. \end{cases} \tag{4.1}$$

$$\begin{cases} u^k\,(t) = K_2(t; x^{k-1}, u^{k-1}), \quad k = 1,2,3,... \\ u^0\,(t) = 0, \quad t \in [0,T]. \end{cases} \tag{4.2}$$

Proof. Since for the zeroth approximation of the iterative process (4.1) the estimate $\left\|\, x^0(t)\,\right\| \le |\, A\,| < \infty$, is valid, then, taking into account the conditions of the theorem, for the first difference of the approximations (4.1) the following estimate is valid

$$\|\, x^1(t) - x^0(t)\,\|_{C[0,T]} \le \frac{1}{\Gamma(\alpha)} \max_{0\le t\le T} \int_0^t (t-s)^{\alpha-1} \left|\, g(s, x^0(s))\,\right| \times$$

$$\times \int_0^s \left|\, f\left(\theta, x^0(\theta), u^0(\theta)\right)\,\right| d\theta ds \le \frac{M_{0,f} T^{\alpha+1}}{\alpha(1+\alpha)\Gamma(\alpha)} = \delta_1 < \infty. \tag{4.3}$$

Since the zeroth approximation of the iterative process (4.2) is equal to zero $\left\| u^0(t) \right\| = 0$, then, taking into account the first and second conditions of the theorem, for the first difference of approximations (4.2) we obtain the following estimate

$$\left\| u^1(t) - u^0(t) \right\| \leq M_g M_{1,f} \max_{0\leq t\leq T} |a_0(t)| \int_0^t (1-s)^{\alpha-1} \int_0^s d\theta ds \leq$$

$$\leq M_g M_{1,f} \max_{0\leq t\leq T} | a_0(t) | \frac{\left| 1-(1+\alpha)(1-t)^{\alpha} + \alpha(1-t)^{\alpha+1} \right|}{\alpha(1+\alpha)} = \delta_2 < \infty. \tag{4.4}$$

We use the first three conditions of the theorem. In order to prove that the operators on the right-hand sides of (2.3) and (3.8) is contracting, for arbitrary positive integer k from approximations (4.1) and (4.2) we obtain

$$\left\| x^{k+1}(t) - x^k(t) \right\|_{C[0,T]} \leq \frac{1}{\Gamma(\alpha)} \max_{0\leq t\leq T} \int_0^t (t-s)^{\alpha-1} \left\| g(s, x^k(s)) \right\|_{C[0,T]} \times$$

$$\times \int_0^s L_2(\theta) \left[\left\| x^k(\theta) - x^{k-1}(\theta) \right\|_{C[0,T]} + \left\| u^k(\theta) - u^{k-1}(\theta) \right\|_{C[0,T]} \right] d\theta ds +$$

$$+ \frac{1}{\Gamma(\alpha)} \max_{0\leq t\leq T} \int_0^t (t-s)^{\alpha-1} L_1(s) \left\| x^k(s) - x^{k-1}(s) \right\| \int_0^s \left\| f(\theta, x^{k-1}(\theta), u^{k-1}(\theta)) \right\| d\theta ds \leq$$

$$\leq \frac{M_g}{\Gamma(\alpha)} \max_{0\leq t\leq T} \int_0^t (t-s)^{\alpha-1} \int_0^s L_2(\theta) d\theta ds \left[\left\| x^k(t) - x^{k-1}(t) \right\| + \left\| u^k(t) - u^{k-1}(t) \right\| \right] +$$

$$+ \frac{M_{0,f}}{\Gamma(\alpha)} \max_{0\leq t\leq T} \int_0^t (t-s)^{\alpha-1} s L_1(s) ds \left\| x^k(t) - x^{k-1}(t) \right\|_{C[0,T]} \leq$$

$$\leq \chi_{11} \left\| x^k(t) - x^{k-1}(t) \right\|_{C[0,T]} + \chi_{12} \left\| u^k(t) - u^{k-1}(t) \right\|_{C[0,T]},$$

where

$$\chi_{11} = \chi_{12} + \frac{M_{0,f}}{\Gamma(\alpha)} \max_{0\leq t\leq T} \int_0^t (t-s)^{\alpha-1} s L_1(s) ds,$$

$$\chi_{12} = \frac{M_g}{\Gamma(\alpha)} \max_{0\leq t\leq T} \int_0^t (t-s)^{\alpha-1} \int_0^s L_2(\theta) d\theta ds.$$

Since $\chi_{11} \geq \chi_{12}$, then we have estimate

$$\left\| x^{k+1}(t) - x^k(t) \right\|_{C[0,T]} \leq$$

$$\leq \chi_{11} \left(\left\| x^k(t) - x^{k-1}(t) \right\|_{C[0,T]} + \left\| u^k(t) - u^{k-1}(t) \right\|_{C[0,T]} \right). \tag{4.5}$$

Similarly, we get the following estimate

$$\left\| u^{k+1}(t) - u^k(t) \right\| \le \frac{1}{\Gamma(\alpha)} \max_{0\le t\le T} \int_0^t (1-s)^{\alpha-1} \left\| g(s, x^k(s)) \right\|_{C[0,T]} \times$$

$$\times \int_0^s L_3(\theta) \left[\left\| x^k(\theta) - x^{k-1}(\theta) \right\|_{C[0,T]} + \left\| u^k(\theta) - u^{k-1}(\theta) \right\|_{C[0,T]} \right] d\theta ds +$$

$$+ \frac{1}{\Gamma(\alpha)} \max_{0\le t\le T} \int_0^t (1-s)^{\alpha-1} L_1(s) \left\| x^k(s) - x^{k-1}(s) \right\| \int_0^s \left\| f_u(\theta, x^{k-1}(\theta), u^{k-1}(\theta)) \right\| d\theta ds \le$$

$$\le \frac{M_g}{\Gamma(\alpha)} \max_{0\le t\le T} \int_0^t (1-s)^{\alpha-1} \int_0^s L_3(\theta) d\theta ds \left[\left\| x^k(t) - x^{k-1}(t) \right\| + \left\| u^k(t) - u^{k-1}(t) \right\| \right] +$$

$$+ \frac{M_{1,f}}{\Gamma(\alpha)} \max_{0\le t\le T} \int_0^t (1-s)^{\alpha-1} s L_1(s) ds \left\| x^k(t) - x^{k-1}(t) \right\|_{C[0,T]} \le$$

$$\le \chi_{21} \left\| x^k(t) - x^{k-1}(t) \right\|_{C[0,T]} + \chi_{22} \left\| u^k(t) - u^{k-1}(t) \right\|_{C[0,T]},$$

where

$$\chi_{21} = \chi_{22} + \frac{M_{1,f}}{\Gamma(\alpha)} \max_{0\le t\le T} \int_0^t (1-s)^{\alpha-1} s L_1(s) ds,$$

$$\chi_{22} = \frac{M_g}{\Gamma(\alpha)} \max_{0\le t\le T} \int_0^t (1-s)^{\alpha-1} \int_0^s L_3(\theta) d\theta ds.$$

Since $\chi_{21} \ge \chi_{22}$, then we have estimate

$$\left\| u^{k+1}(t) - u^k(t) \right\|_{C[0,T]} \le$$

$$\le \chi_{21} \left(\left\| x^k(t) - x^{k-1}(t) \right\|_{C[0,T]} + \left\| u^k(t) - u^{k-1}(t) \right\|_{C[0,T]} \right). \tag{4.6}$$

Adding the estimate (4.5) to estimate (4.6), we obtain that

$$\left\| U^{k+1}(t) - U^k(t) \right\|_{C[0,T]} \le \rho \left\| U^k(t) - U^{k-1}(t) \right\|_{C[0,T]}, \tag{4.7}$$

where $\rho = \chi_{11} + \chi_{21}$,

$$\left\| U^k(t) - U^{k-1}(t) \right\|_{C[0,T]} = \left\| x^k(t) - x^{k-1}(t) \right\|_{C[0,T]} + \left\| u^k(t) - u^{k-1}(t) \right\|_{C[0,T]}.$$

According to the last condition of the theorem, we have $\rho = \chi_{11} + \chi_{21} < 1$, and

$$\left\| x^k(t) - x^{k-1}(t) \right\|_{C[0,T]} \le \left\| U^k(t) - U^{k-1}(t) \right\|_{C[0,T]}, \tag{4.8}$$

$$\left\| u^k(t) - u^{k-1}(t) \right\|_{C[0,T]} \leq \left\| U^k(t) - U^{k-1}(t) \right\|_{C[0,T]}. \tag{4.9}$$

Then from the estimate (4.7) operators on the right-hand sides of (2.3) and (3.8) has a unique pair of fixed points on the interval $[0,T]$. Hence, and from the estimates (4.3), (4.4), (4.7) we deduce that there exists a unique control function $u(t) \in \Upsilon$ and unique state function $x(t) \in C([0,T], \mathbb{R}^n)$. The solution of the integral equation (3.8) we denote by $u(t) = h(t)$. Then, it is easy to prove the solvability of the differential equation (2.4). The theorem 4.1 is proved. □

Theorem 4.2. *Let the conditions of the theorem 4.1 be fulfilled. Then the state vector-function $x(t) \in C([0,T], \mathbb{R}^n)$ is continuously dependence from initial vector A.*

Proof. Now we show the continuous dependence of the solution of the problem (1.1)–(1.2) on the right-hand side of the initial condition (1.2). Let $A_1, A_2 \in \mathbb{R}^n$ be two different vectors and $x_1(t)$, $x_2(t) \in C([0,T], \mathbb{R}^n)$ be the corresponding solutions of the problem (1.1)–(1.2) and $|A_1 - A_2| < \delta$, where $0 < \delta$ is a small number depending on a given small positive number ε. Then, similar to the estimates (4.3) and (4.6) from the equations (2.3) and (3.8) we obtain that

$$\| x_1(t) - x_2(t) \|_{C[0,T]} \leq | A_1 - A_2 | +$$

$$+\chi_{11} \left(\| x_1(t) - x_2(t) \|_{C[0,T]} + \| u_1(t) - u_2(t) \|_{C[0,T]} \right), \tag{4.10}$$

$$\| u_1(t) - u_2(t) \|_{C[0,T]} \leq \chi_{21} \left(\| x_1(t) - x_2(t) \|_{C[0,T]} + \| u_1(t) - u_2(t) \|_{C[0,T]} \right). \tag{4.11}$$

Adding the estimates (4.10) and (4.11) term by term, we obtain that the estimate is valid

$$\| U_1(t) - U_2(t) \|_{C[0,T]} \leq | A_1 - A_2 | + \rho \| U_1(t) - U_2(t) \|_{C[0,T]}, \tag{4.12}$$

where

$$\| U_1(t) - U_2(t) \|_{C[0,T]} = \| x_1(t) - x_2(t) \|_{C[0,T]} + \| u_1(t) - u_2(t) \|_{C[0,T]}.$$

From the estimate (4.12) we derive

$$\|U_1(t) - U_2(t)\|_{C[0,T]} \leq | A_1 - A_2 | + \rho \cdot \|U_1(t) - U_2(t)\|_{C[0,T]}. \tag{4.13}$$

According to the last condition of the theorem, we have $\rho < 1$. Therefore, from the inequality (4.13) it follows that

$$\| U_1(t) - U_2(t) \|_{C[0,T]} \leq (1-\rho)^{-1} | A_1 - A_2 |.$$

Since $| A_1 - A_2 | < \delta$, from the last estimate we obtain

$$\| U_1(t) - U_2(t) \|_{C[0,T]} < \varepsilon,$$

where $\varepsilon = \delta(1-\rho)^{-1}$. Estimates similar to those of (4.8) and (4.9) are also valid here. Therefore, we have

$$\| x_1(t) - x_2(t) \|_{C[0,T]} < \varepsilon.$$

The theorem 4.2 is proved □

Conclusion

In this paper, an optimal control problem for a nonlinear system of fractional-order integro-differential equations involving the Gerasimov–Caputo operator has been investigated. The considered model combines several analytically difficult features simultaneously: fractional memory effects, nonlinear integral interactions, and the product structure of nonlinear functions.

A distinctive feature of the proposed approach is that the state function and the control function are constructed simultaneously as a coupled solution pair of a nonlinear functional-integral system. This differs substantially from the classical approach in optimal control theory, where the control function is usually treated as known during the construction of the state trajectory.

Using fractional integration techniques, the original integro-differential equation was transformed into an equivalent nonlinear integral equation for the state function. By introducing an adjoint final-value problem and constructing the Pontryagin function, an additional nonlinear integral equation for the control function was obtained. As a result, the original optimal control problem was reduced to a strongly coupled nonlinear system.

A constructive analytical method based on successive approximations and the contraction mapping principle was developed. Explicit sufficient conditions guaranteeing existence and uniqueness of the vector-valued control function and the corresponding state trajectory were established. In addition, continuous dependence of the solution on the initial vector was proved, which provides an important stability property of the considered optimal control system.

The obtained results extend several known solvability results for nonlinear fractional dynamical systems and provide a constructive framework for studying fractional optimal control problems with memory effects and nonlinear nonlocal interactions.

The proposed methods can be further generalized to broader classes of systems, including impulsive fractional equations, systems with delays and maxima, distributed parameter models, nonlocal boundary-value problems, and fractional systems in locally convex or infinite-dimensional functional spaces.

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