ISSN: 1694-9242
Journal of Osh University "Differential Equations", 2026, Volume 1, Issue 2, P. 11–24.
https://doi.org/10.52754/16949242_2026_2_2

UDC 517.929

Periodic Solutions of a System of Nonlinear Integro-Differential Equations with a Product of Two Nonlinear Functions and Maxima

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Received: February 03, 2026; **First on-line published**: June 25, 2026

Abstract. *This paper investigates periodic solutions of a nonlinear fractional-order integro-differential system involving the Gerasimov–Caputo operator, a product of two nonlinear functions, and maxima-type nonlinearities. The considered model combines hereditary effects, nonlinear memory interactions, and delayed maximal deviations, which significantly complicate the qualitative analysis of the system.*
The problem is studied under periodic boundary conditions in a Banach space framework. By applying fractional integration techniques, the original nonlinear fractional integro-differential system is reduced to an equivalent nonlinear functional-integral equation with maxima.
A constructive analytical approach based on the method of successive approximations and the contraction mapping principle is developed. Explicit sufficient conditions ensuring existence and uniqueness of periodic solutions are obtained. In addition, iterative schemes allowing the practical computation of periodic solutions are constructed.
The existence problem for periodic solutions is further reduced to the investigation of zeros of a nonlinear operator field generated by the associated integral system. Using homotopy arguments and topological index methods, sufficient conditions for the existence of periodic solutions are established.
The obtained results extend several known solvability results for nonlinear fractional dynamical systems with delays and maxima and provide a constructive framework for studying periodic processes with memory and nonlocal interactions.

Key words: *Periodic solutions, integro-differential equations, product of two nonlinear functions, method of successive approximations, contraction mapping, existence, uniqueness.*

Introduction

Fractional differential and integro-differential equations have become an important mathematical instrument for describing systems with hereditary properties, memory effects, anomalous diffusion, and nonlocal interactions. Such models naturally arise in viscoelasticity, continuum mechanics, biological systems, signal processing, heat transfer, control theory, and many other branches of applied science. A lot of publications of studying on differential equations with impulsive effects, describing many natural and technical processes, are appearing [1]–[3].

In recent years, substantial attention has been devoted to the study of periodic solutions of nonlinear dynamical systems. Periodic regimes play a fundamental role in the qualitative theory of differential equations because they describe recurrent processes, oscillatory phenomena, and stable long-term behaviors of complex systems In recent years the interest in the study of differential equations with periodical boundary conditions has increased. In the works [4, 5, 6, 7, 8] periodic solutions of the differential equations with impulsive effects are studied. In [9] the problems of bifurcation of positive periodic solutions of first-order impulsive differential equations are studied. In [10] the problems of stability of periodic impulsive systems are studied. In many practical applications, the presence of memory effects and delayed interactions leads naturally to mathematical models involving fractional operators and maxima-type nonlinearities.

Differential equations with maxima constitute a special class of nonlinear functional equations in which the future dynamics depend not only on the current state but also on extremal values of the state over a preceding interval. Such equations arise in automatic control systems, engineering processes with threshold mechanisms, biological models with delayed feedback, and optimization problems involving extremal responses. In this paper, in contrast to works [4]–[10], we study a periodical boundary value problem for a system of first order differential equations with impulsive effects and "maxima". The questions of the existence and uniqueness of the solution to the periodical boundary value problem are studied. In addition, the technique used in our work is constructive (see, [11]–[14]) and allows the practical calculation of periodic solutions of nonlinear dynamical systems with deviations, including those with maxima.

At the same time, fractional operators substantially increase the analytical complexity of the problem due to their intrinsic nonlocal structure. The combination of fractional memory effects, nonlinear integral interactions, and maxima-type nonlinearities leads to highly nontrivial mathematical models whose qualitative analysis remains insufficiently developed [15]–[44].

Most existing studies on periodic problems for nonlinear systems focus either on classical differential equations or on fractional equations without maxima and nonlinear product structures. The simultaneous presence of a Gerasimov–Caputo fractional operator, nonlinear Volterra-type interactions, and maxima terms creates additional analytical difficulties related to compactness, nonlinear estimates, and the construction of periodic trajectories.

Motivated by these considerations, in the present paper we investigate periodic solutions of a nonlinear fractional-order integro-differential system involving a product of two nonlinear functions and maxima. The problem is considered under periodic boundary conditions.

The main contributions of the paper are the following:

- reduction of the original fractional integro-differential system to an equivalent nonlinear functional-integral equation with maxima;

- derivation of explicit estimates for the associated nonlinear integral operator;

- development of a constructive iterative scheme for approximating periodic solutions;
- proof of existence and uniqueness of periodic solutions using the contraction mapping principle;
- reduction of the periodic solvability problem to the investigation of zeros of a nonlinear operator field;
- application of homotopy and topological index techniques to establish existence of periodic trajectories.

The developed approach provides an effective analytical framework for studying nonlinear periodic processes with memory, delay, and maxima-type interactions in fractional dynamical systems.

1 Problem statement

In this paper we use the Gerasimov–Caputo fractional differential operator defined by the formula

$$ {}_C D_{0t}^{\alpha} x(t) = I_{0t}^{1-\alpha} x'(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{x'(s)ds}{(t-s)^{\alpha}}, \ \ t \in (0,T), $$

where

$$ I_{0t}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} x(s) ds, \quad 0 < \alpha \le 1 $$

is Riemann–Liouville integral operator, α is order of fractional differential operator, $\Gamma(\alpha)$ is Gamma function.

On the interval $[0,T]$ we consider the questions of existence and constructive methods of finding the periodic solutions of the system of nonlinear ordinary fractional order integro-differential equations with maxima

$$ {}_C D_{0t}^{\alpha} x(t) = \int\limits_0^t g\left(s, x(s)\right) f\left(s, x(s), \max\left\{x(\tau) : \tau \in P(s)\right\}\right) ds, \tag{1.1} $$

where $x, y \in X$, X is the closed bounded domain in the space $\mathbb{R}^n$, ∂X is its border, $g \in C([0,T] \times X, \mathbb{R}^n)$, $f \in C([0,T] \times X \times X, \mathbb{R}^n)$, functions g, f are $T-$periodic, $P(t) = [t-h, t]$, $0 < h = \text{const}$.

This equation (1.1) we study with periodic conditions

$$ \begin{cases} x(t) = \varphi(t), \quad t \in [-h, 0], \\ x(0) = x(T), \end{cases} \tag{1.2} $$

where $\varphi(t) \in C[-h, 0]$.

By $C\left([0,T], \mathbb{R}^n\right)$ denoted the Banach space, which consists of continuous vector functions $x(t)$, defined on the segment $[0,T]$, with values in $\mathbb{R}^n$ and with the norm

$$ \| \, x \, \|_{C[0,T]} = \sqrt{\sum_{j=1}^{n} \max_{0 \le t \le T} | \, x_j(t) \, |^2}. $$

By the $BD\left([0,T],\mathbb{R}^n\right)$ we denote the Banach space of continuous functions on the interval $[0,T]$ with the norm

$$\| x(t) \|_{BD[0,T]} = \| x(t) \|_{C[0,T]} + h \cdot \| x'(t) \|_{C[0,T]} .$$

Problem 1.1 (Formulation of problem)**.** To find the $T-$periodic function $x(t) \in BD\left([0,T],\mathbb{R}^n\right)$, which for all $t \in [0,T]$ satisfies the system of integro-differential equation (1.1), conditions (1.2) and goes through x_0 at $t=0$.

2 Reduction to a functional-integral equation

Let the function $x(t) \in BD\left([0,T],\mathbb{R}^n\right)$ is a solution of the periodic boundary value problem (1.1), (1.2). We act the operator $I_{0\,t}^{\alpha}$ to the equation (1.1)

$$I_{0\,t\,C}^{\alpha}D_{0t}^{\alpha}x(t) = I_{0\,t}^{\alpha}\int\limits_0^t g\left(s,x(s)\right) f\left(s,x(s),\max\left\{x(\tau):\tau\in P(s)\right\}\right)ds.$$

Using the definition of the operator, we integrate of the equation (1.1) on the given interval $(0,t)$

$$I_{0\,t\,C}^{\alpha}D_{0\,t}^{\alpha}\left(x(t)\right) = I_{0\,t}^{\alpha}I_{0\,t}^{1-\alpha}x'(t) = I_{0\,t}^{1}x'(t) = \int\limits_0^t x'(s)ds = x(t)-x(0), \tag{2.1}$$

$$I_{0t}^{\alpha}\eta\left(t,x(t)\right) = \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\eta\left(s,x(s)\right)ds. \tag{2.2}$$

Appling the formulas (2.1) and (2.2) to the last fractional integro-differential equation, we obtain

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\int\limits_0^s g(\theta,x)f(\theta,x,y)d\theta ds, \tag{2.3}$$

where $g(t,x)=g(t,x(t)), \quad f(t,x,y) = f\left(t,x(t),\max\left\{x(\tau):\tau\in P(t)\right\}\right).$

We subordinate the function $x(t) \in BD\left([0,T],\mathbb{R}^n\right)$ in (2.3) to satisfy the periodic condition in (1.2). In this order, from (2.3) we have

$$x(T) = x(0) + \frac{1}{\Gamma(\alpha)}\int\limits_0^T (T-s)^{\alpha-1}\int\limits_0^s g(\theta,x)f(\theta,x,y)d\theta ds.$$

Hence, taking the condition (1.2) into account, we obtain:

$$\int\limits_0^T (T-s)^{\alpha-1}\int\limits_0^s g(\theta,x)f(\theta,x,y)d\theta ds = 0.$$

Using the zero equality, the integro-differential equation (1.1) one can write as

$${}_{C}D_{0t}^{\alpha}x(t) = \int\limits_0^t g\left(s,x(s)\right) f\left(s,x(s),\max\left\{x(\tau):\tau\in P(s)\right\}\right)ds-$$

$$-\frac{1}{T}\int\limits_0^T (T-s)^{\alpha-1}\int\limits_0^s g\left(\theta, x(\theta)\right) f\left(\theta, x(\theta), \max\left\{x(\tau) : \tau \in P(\theta)\right\}\right) d\theta ds. \tag{2.4}$$

Then, by integration of the equation (2.4) on the interval $(0, t)$, we obtain the following functional-integral equation with maxima:

$$x(t) = x_0 +$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\left[\int\limits_0^s g\left(\theta, x(\theta)\right) f\left(\theta, x(\theta), \max\left\{x(\tau) : \tau \in P(\theta)\right\}\right) d\theta - \right.$$

$$\left. -\frac{1}{T}\int\limits_0^T (T-\theta)^{\alpha-1}\int\limits_0^\theta g\left(\varsigma, x(\varsigma)\right) f\left(\varsigma, x(\varsigma), \max\left\{x(\tau) : \tau \in P(\varsigma)\right\}\right) d\varsigma d\theta\right] ds. \tag{2.5}$$

Lemma 2.1. *For the equation* (2.5) *is true the following estimate*

$$\| x(t) - x_0 \|_{C[0,T]} \leq \frac{T^{\alpha+1}}{\Gamma(2+\alpha)} M_g M_f, \tag{2.6}$$

where $M_g = \| g\left(t, x(t)\right) \|_{C[0,T]}\,, \quad M_f = \| f\left(t, x(t), y(t)\right) \|_{C[0,T]}\,.$

Proof. We rewrite the integral equation (2.5) as

$$x(t) - x_0 =$$

$$= \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\int\limits_0^s g\left(\theta, x(\theta)\right) f\left(\theta, x(\theta), \max\left\{x(\tau) : \tau \in P(\theta)\right\}\right) d\theta ds -$$

$$-\frac{1}{T}\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1}\int\limits_0^T (T-\theta)^{\alpha-1}\int\limits_0^\theta g\left(\varsigma, x(\varsigma)\right) f\left(\varsigma, x(\varsigma), \max\left\{x(\tau) : \tau \in P(\varsigma)\right\}\right) d\varsigma d\theta ds.$$

Hence, we obtain the following estimate

$$\| x(t) - x_0 \|_{C[0,T]} \leq \frac{\alpha(t)}{\Gamma(\alpha)} \| g\left(t, x(t)\right) \|_{C[0,T]} \| f\left(t, x(t), y(t)\right) \|_{C[0,T]}\,, \tag{2.7}$$

where

$$\alpha(t) = \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1} s ds - \frac{1}{\Gamma(\alpha)}\frac{1}{T}\int\limits_0^t (t-s)^{\alpha-1}\int\limits_0^T (T-\theta)^{\alpha-1}\theta d\theta ds.$$

We calculate the integrals:

$$A_1 = \int\limits_0^t (t-s)^{\alpha-1} s ds = \left| \begin{array}{l} s = tz \\ s = 0 \Rightarrow z = 0 \\ s = t \Rightarrow z = 1 \end{array} \right| = \int\limits_0^1 t^{\alpha-1}(1-z)^{\alpha-1} tz t dz = t^{\alpha+1} B(2, \alpha).$$

Similarly, we obtain that

$$A_2 = \int\limits_0^T (T-s)^{\alpha-1} s ds = T^{\alpha+1} B(2, \alpha),$$

$$A_3 = \int_0^t (t-s)^{\alpha-1} \int_0^T (T-\theta)^{\alpha-1}\theta d\theta ds = A_2 \int_0^t (t-s)^{\alpha-1} ds = T^{2\alpha+1}\frac{\Gamma(\alpha)}{\Gamma(2+\alpha)}.$$

Then we have the following estimate

$$\alpha(t) = \frac{1}{\Gamma(\alpha)}A_1 - \frac{1}{\Gamma(\alpha)}\frac{1}{T}A_3 = \frac{1}{\Gamma(\alpha)}t^{\alpha+1}\frac{\Gamma(\alpha)}{\Gamma(2+\alpha)} - \frac{1}{\Gamma(\alpha)}\frac{1}{T}T^{2\alpha+1}\frac{\Gamma(\alpha)}{\Gamma(2+\alpha)} =$$

$$= \frac{t^{\alpha+1}}{\Gamma(2+\alpha)} - \frac{T^{2\alpha}}{\Gamma(2+\alpha)} \le \frac{T^{\alpha+1}}{\Gamma(2+\alpha)} = \frac{T^{\alpha+1}}{\alpha(1+\alpha)\Gamma(\alpha)}.$$

Now, it is easy to check, that from (2.7) follows (2.6). Lemma 2.1 is proved. □

Remark 2.1. $T-$periodic solution $x_{\varphi(t)} = \psi(t)$ of the system (**??**) with initial value function $\varphi(t)$ on the initial set $[-h,0]$ is defined by the initial value function $\varphi(t)$, which is periodical continuation of the solution $\psi(t)$ into initial set $[-h,0]$.

By differentiating integral equation (2.5), we obtain

$$x'(t) =$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^t (1-s)^{\alpha-1}\left[\int_0^s g\left(\theta, x(\theta)\right) f\left(\theta, x(\theta), \max\left\{x(\tau) : \tau \in P(\theta)\right\}\right) d\theta - \right.$$

$$\left. - \frac{1}{T}\int_0^T (T-\theta)^{\alpha-1}\int_0^\theta g\left(\varsigma, x(\varsigma)\right) f\left(\varsigma, x(\varsigma), \max\left\{x(\tau) : \tau \in P(\varsigma)\right\}\right) d\varsigma d\theta\right] ds. \quad (2.8)$$

To obtain estimates we use the following lemma.

Lemma 2.2. *For the difference of two functions with maxima there holds the following estimate*

$$\|\max\left\{x(\tau)\,|\tau \in P(t)\right\} - \max\left\{y(\tau)\,|\tau \in P(t)\right\}\|_{C[0,T]} \le$$

$$\le \|x(t) - y(t)\|_{C[0,T]} + h\left\|\frac{\partial}{\partial t}\left[x(t) - y(t)\right]\right\|_{C[0,T]}.$$

Theorem 2.1. *Assume that for all $t \in [0,T]$ the following conditions are fulfilled:* 1). $\|g(t,x(t))\|_{C[0,T]} \le M_g < \infty$; $\|f(t,x(t),y(t))\|_{C[0,T]} \le M_f < \infty$;

2). $\|g(t,x_1) - g(t,x_2)\|_{C[0,T]} \le L_1\|x_1 - x_2\|_{C[0,T]}$;

3). $\|f(t,x_1,y_1) - f(t,x_2,y_2)\|_{C[0,T]} \le L_2\left[\|x_1 - x_2\|_{C[0,T]} + \|y_1 - y_2\|_{C[0,T]}\right]$;

4). *The radius of the inscribed ball in X is greater than* $\frac{T^{\alpha+1}}{\Gamma(2+\alpha)}M_g M_f$;

5). $\rho = 2(M_f L_1 + M_g L_2)\left(\frac{T^{\alpha+1}}{\Gamma(2+\alpha)} + h\alpha_0'\right) < 1$.

If the system (1.1) *has a periodic solution for all $t \in [0,T]$, then this solution can be founded by the system of nonlinear functional-integral equations*

$$x(t,x_0) = x_0 +$$

$$+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[\int_0^s g\left(\theta, x(\theta,x_0)\right) f\left(\theta, x(\theta,x_0), \max\left\{x(\tau,x_0) : \tau \in P(\theta)\right\}\right) d\theta - \right.$$

$$-\frac{1}{T}\int_0^T (T-\theta)^{\alpha-1}\int_0^\theta g\left(\varsigma, x(\varsigma,x_0)\right) f\left(\varsigma, x(\varsigma,x_0), \max\left\{x(\tau,x_0):\tau\in P(\varsigma)\right\}\right) d\varsigma d\theta\Bigg] ds. \qquad (2.9)$$

Proof. We will show that the right-hand side of the system of equations (2.9) as an operator takes a ball with radius $\frac{T^{\alpha+1}}{\Gamma(2+\alpha)}M_gM_f$ into itself and is a contraction operator. So, according to the lemma 2.1, from (2.9) we have

$$\| x(t,x_0)-x_0 \|_{C[0,T]} \le \frac{T^{\alpha+1}}{\Gamma(2+\alpha)}M_gM_f. \qquad (2.10)$$

From the equation (2.8) we obtain

$$\| x'(t,x_0) \|_{C[0,T]} \le 2\alpha(t)M_gM_f, \qquad (2.11)$$

where

$$\alpha'(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (1-s)^{\alpha-1}sds-\frac{1}{\Gamma(\alpha)}\frac{1}{T}\int_0^t (1-s)^{\alpha-1}\int_0^T (T-\theta)^{\alpha-1}\theta d\theta ds.$$

We calculate the integrals:

$$B_1=\int_0^t (1-s)^{\alpha-1}sds=\frac{1-(1+\alpha)(1-t)^\alpha+\alpha(1-t)^{\alpha+1}}{\alpha(1+\alpha)},$$

Similarly, we obtain that

$$B_2=\int_0^T (T-s)^{\alpha-1}sds=T^{\alpha+1}B(2,\alpha)=T^{\alpha+1}\frac{\Gamma(\alpha)}{\Gamma(2+\alpha)}=\frac{T^{\alpha+1}}{\alpha(1+\alpha)},$$

$$B_3=\int_0^t (1-s)^{\alpha-1}ds\int_0^T (T-\theta)^{\alpha-1}\theta d\theta=B_2\int_0^t (1-s)^{\alpha-1}ds=$$

$$=\frac{T^{\alpha+1}}{\alpha(1+\alpha)}\int_0^t (t-s)^{\alpha-1}ds=\frac{T^{2\alpha+1}}{\alpha^2(1+\alpha)}.$$

Then we have

$$\alpha'(t)=\frac{1}{\Gamma(\alpha)}B_1-\frac{1}{\Gamma(\alpha)}\frac{1}{T}B_3=\frac{1-(1+\alpha)(1-t)^\alpha+\alpha(1-t)^{\alpha+1}}{\alpha(1+\alpha)\Gamma(\alpha)}-$$

$$-\frac{1}{\Gamma(\alpha)}\frac{T^{2\alpha}}{\alpha^2(1+\alpha)}\le\frac{1-(1+\alpha)(1-t)^\alpha+(1-t)^{\alpha+1}}{\alpha(1+\alpha)\Gamma(\alpha)}\le\alpha_0',$$

where

$$\alpha_0'=\begin{cases}\dfrac{1+\alpha(1-t)^{\alpha+1}}{\alpha(1+\alpha)\Gamma(\alpha)}, & T<1,\\ \dfrac{1}{\alpha(1+\alpha)\Gamma(\alpha)}, & T=1,\\ \dfrac{1-(1+\alpha)(1-t)^\alpha}{\alpha(1+\alpha)\Gamma(\alpha)}, & T>1.\end{cases}$$

We consider a difference $x(t,x_0)-\vartheta(t,x_0)$, where the functions $x(t,x_0)$ and $\vartheta(t,x_0)$ are satisfy the system of equations (2.6). By virtue of lemma 2 and the conditions of the theorem, from (2.9) we have

$$\| x(t,x_0)-\vartheta(t,x_0) \|_{C[0,T]} \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} \left\{ \int\limits_0^s \Big[(M_f L_1 + M_g L_2) \| x(\theta,x_0)-\vartheta(\theta,x_0) \|_{C[0,T]} + \right.$$

$$+ M_g L_2 \| \max\{x(\tau,x_0) : \tau \in P(\theta)\} - \max\{\vartheta(\tau,x_0) : \tau \in P(\theta)\} \|_{C[0,T]} \Big] d\theta +$$

$$+ \frac{1}{T} \int\limits_0^T (T-\theta)^{\alpha-1} \int\limits_0^\theta \Big[(M_f L_1 + M_g L_2) \| x(\theta,x_0)-\vartheta(\theta,x_0) \|_{C[0,T]} +$$

$$\left. + M_g L_2 \| \max\{x(\tau,x_0) : \tau \in P(\varsigma)\} - \max\{\vartheta(\tau,x_0) : \tau \in P(\varsigma)\} \|_{C[0,T]} \Big] d\varsigma d\theta \right\} ds \leq$$

$$\leq \frac{T^{\alpha+1}}{\Gamma(2+\alpha)} \Big[(M_f L_1 + M_g L_2) \| x(t,x_0)-\vartheta(t,x_0) \|_{C[0,T]} +$$

$$+ M_g L_2 h \left\| x'(t,x_0)-\vartheta'(t,x_0) \right\|_{C[0,T]} \Big]. \tag{2.12}$$

Similarly, by the assumptions of the theorem 2.1, from (2.8) we have

$$\left\| x'(t,x_0)-\vartheta'(t,x_0) \right\|_{C[0,T]} \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \int\limits_0^t (1-s)^{\alpha-1} \left\{ \int\limits_0^s \Big[(M_f L_1 + M_g L_2) \| x(\theta,x_0)-\vartheta(\theta,x_0) \|_{C[0,T]} + \right.$$

$$+ M_g L_2 \| \max\{x(\tau,x_0) : \tau \in P(\theta)\} - \max\{\vartheta(\tau,x_0) : \tau \in P(\theta)\} \|_{C[0,T]} \Big] d\theta +$$

$$+ \frac{1}{T} \int\limits_0^T (T-\theta)^{\alpha-1} \int\limits_0^\theta \Big[(M_f L_1 + M_g L_2) \| x(\theta,x_0)-\vartheta(\theta,x_0) \|_{C[0,T]} +$$

$$\left. + M_g L_2 \| \max\{x(\tau,x_0) : \tau \in P(\varsigma)\} - \max\{\vartheta(\tau,x_0) : \tau \in P(\varsigma)\} \|_{C[0,T]} \Big] d\varsigma d\theta \right\} ds \leq$$

$$\leq 2\alpha_0' \Big[(M_f L_1 + M_g L_2) \| x(t,x_0)-\vartheta(t,x_0) \|_{C[0,T]} +$$

$$+ M_g L_2 h \left\| x'(t,x_0)-\vartheta'(t,x_0) \right\|_{C[0,T]} \Big]. \tag{2.13}$$

So, we have two inequalities: (2.12) and (2.13)

$$\| x(t,x_0)-\vartheta(t,x_0) \|_{C[0,T]} \leq$$

$$\leq \frac{2T^{\alpha+1}}{\Gamma(2+\alpha)} \Big[(M_f L_1 + M_g L_2) \| x(t,x_0)-\vartheta(t,x_0) \|_{C[0,T]} +$$

$$+ M_g L_2 h \ \left\| x'(t,x_0)-\vartheta'(t,x_0) \right\|_{C[0,T]} \Big], \tag{2.14}$$

$$\left\| x'(t,x_0)-\vartheta'(t,x_0) \right\| \leq$$

$$\leq 2\alpha_0' \left[(M_f L_1 + M_g L_2) \| x(t,x_0) - \vartheta(t,x_0) \|_{C[0,T]} + \right.$$

$$\left. + M_g L_2 h \ \left\| x'(t,x_0) - \vartheta'(t,x_0) \right\|_{C[0,T]} \right]. \tag{2.15}$$

Multiplying both sides of (2.15) to h and the result adding to (2.14), we obtain

$$\| x(t,x_0) - \vartheta(t,x_0) \|_{BD[0,T]} \leq \rho \cdot \| x(t,x_0) - \vartheta(t,x_0) \|_{BD[0,T]}, \tag{2.16}$$

where

$$\rho = 2(M_f L_1 + M_g L_2) \left(\frac{T^{\alpha+1}}{\Gamma(2+\alpha)} + h\alpha_0' \right).$$

According to the last condition of the theorem 2.1, $\rho < 1$. So, from the estimate (2.16) we deduce that the operator on right-hand side of (2.9) is contracting. From the estimates (2.10), (2.11), (2.14) and (2.15) implies that there exists unique fixed point $x(t,x_0) \in BD[0,T]$. So, the theorem 2.1 is proved. □

From the estimate (2.16) it is easy to obtain that for x_0, $\bar{x}_0 \in X$ holds

$$\| x(t,x_0) - x(t,\bar{x}_0) \|_{BD[0,T]} \leq \frac{|\, x_0 - \bar{x}_0 \,|}{1-\rho}.$$

We note that the theorem 1 one can proof by the method of successive approximations, defining iteration process as

$$x_0(t,x_0) = x_0, \ \ x_{k+1}(t,x_0) = x_0 +$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\int_0^s g\left(\theta, x_k(\theta,x_0)\right) f\left(\theta, x_k(\theta,x_0), \max\left\{x_k(\tau,x_0) : \tau \in P(\theta)\right\}\right) d\theta - \right.$$

$$\left. - \frac{1}{T} \int_0^T (T-\theta)^{\alpha-1} \int_0^\theta g\left(\varsigma, x_k(\varsigma,x_0)\right) f\left(\varsigma, x_k(\varsigma,x_0), \max\left\{x_k(\tau,x_0) : \tau \in P(\varsigma)\right\}\right) d\varsigma d\theta \right] ds.$$

(**??**)

Now we will show the existence of periodic solutions of the system of integro-differential equations (1.1). We introduce the following designations:

$$\Delta(x_0) = \frac{1}{T} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times$$

$$\times \int_0^T (T-\theta)^{\alpha-1} \int_0^\theta g\left(\varsigma, x_\infty(\varsigma,x_0)\right) f\left(\varsigma, x_\infty(\varsigma,x_0), \max\left\{x_\infty(\tau,x_0) : \tau \in P(\varsigma)\right\}\right) d\varsigma d\theta ds, \tag{2.17}$$

$$\Delta_k(x_0) = \frac{1}{T} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times$$

$$\times \int_0^T (T-\theta)^{\alpha-1} \int_0^\theta g\left(\varsigma, x_k(\varsigma,x_0)\right) f\left(\varsigma, x_k(\varsigma,x_0), \max\left\{x_k(\tau,x_0) : \tau \in P(\varsigma)\right\}\right) d\varsigma d\theta ds, \tag{2.18}$$

where $x(t,x_0) = \lim\limits_{k\to\infty} x_k(t,x_0) = x_\infty(t,x_0)$ is solution of nonlinear system of integral (2.9). Therefore, $x_\infty(t,x_0)$ is the solution of the system of integro-differential equations (1.1) for $\Delta(x_0) = 0$ going through x_0 at $t = 0$. Consequently, the questions of existence of solution of the system of integro-differential equations (1.1) we reduce to the questions of existence of zeros of function $\Delta(x_0)$ and we solve this problem finding zeros of the iteration $\Delta_k(x_0)$.

Theorem 2.2. *Assume that*
1). *All conditions of the theorem 2.1 are fulfilled;*
2). *There is a natural number k such that the function $\Delta_k(x_0)$ has an isolated singular point $\Delta_k(x_0) = 0$, index of which is nonzero;*
3). *There is a closed convex region $X_0 \subset X$, containing a single singular point such that on the its border ∂X_0 is fulfilled estimate*

$$\inf_{x\in\partial X_0} \| \Delta_k(x) \|_{BD[0,T]} \geq M_g M_f \frac{\rho^{k+1}}{1-\rho}. \tag{2.19}$$

Then the system of integro-differential equations (1.1) *has a periodic solution for all $t \in [0,T]$ that $x(0) \in X_0$.*

Proof. Let us consider families of everywhere continuous on ∂X vector fields

$$V(\sigma, x_0) = \Delta_k(x_0) + \sigma(\Delta(x_0) - \Delta_k(x_0)),$$

which connect the two fields

$$V(0,x_0) = \Delta_k(x_0), V(1,x_0) = \Delta(x_0).$$

We note that there is true the estimate

$$\| \Delta(x_0) - \Delta_k(x_0) \|_{BD[0,T]} \leq M_g M_f \frac{\rho^{k+1}}{1-\rho}. \tag{2.20}$$

Therefore, the vector field $V(\sigma,x_0)$ does not vanish anywhere on ∂X_0. Indeed, from (**??**) and (2.20) implies that

$$\| V(\sigma,x_0) \|_{BD[0,T]} \geq \| \Delta_k(x_0) \|_{BD[0,T]} - \| \Delta(x_0) - \Delta_k(x_0) \|_{BD[0,T]} > 0. \tag{2.21}$$

The fields $\Delta_k(x_0)$ and $\Delta(x_0)$ are homotopic on ∂X and the rotations of the fields homotopic on the compact are equal. Hence, taking into account (2.21), we conclude that the rotation of the field $\Delta(x_0)$ on the ∂X_0 is equal to the index of the singular point x_0 of the field $\Delta_k(x_0)$ and nonzero. Consequently, the vector field $\Delta(x_0)$ on the X_0 has a singular point x_0, for which $\Delta(x_0) = 0$. Therefore, the system of impulsive differential equations (1.1) has a periodic solution for all $t \in [0,T]$, that $x(0) \in X_0$. In addition, we note that for x_0, $\bar{x}_0 \in X$ from (2.14) and (2.15) we have

$$\| \Delta(x_0) \|_{BD[0,T]} \leq M_g M_f,$$

$$\| \Delta(x_0) - \Delta(\bar{x}_0) \|_{BD[0,T]} \leq \frac{|x_0 - \bar{x}_0|}{1-\rho}.$$

Theorem is proved. □

Conclusion

In this paper, periodic solutions of a nonlinear fractional-order integro-differential system involving the Gerasimov–Caputo operator, maxima-type nonlinearities, and a product of two nonlinear functions have been investigated. The considered model combines several analytically difficult features simultaneously: fractional memory effects, nonlinear integral interactions, delayed maximal deviations, and periodic boundary conditions.

By applying fractional integration techniques, the original nonlinear system was transformed into an equivalent functional-integral equation with maxima. This reduction made it possible to construct a constructive analytical framework for studying periodic solutions of the problem.

Using the method of successive approximations together with the contraction mapping principle, sufficient conditions guaranteeing existence and uniqueness of periodic solutions were obtained. Explicit estimates for the associated nonlinear operators were derived, which allowed the construction of convergent iterative procedures for practical approximation of periodic trajectories.

A further important aspect of the work is the reduction of the periodic solvability problem to the investigation of zeros of a nonlinear operator field generated by the integral equation. By combining homotopy arguments with topological index methods, sufficient conditions for the existence of periodic solutions were established.

The obtained results generalize several known solvability results for nonlinear fractional systems with delays and maxima and provide a constructive methodology for studying periodic regimes in systems with memory and nonlocal interactions.

The proposed approach can be extended to broader classes of fractional dynamical systems, including impulsive systems, equations with nonlocal boundary conditions, distributed parameter models, and fractional systems in infinite-dimensional functional spaces. The methods developed in this work may also be useful for applications in nonlinear control theory, hereditary mechanics, biological oscillation models, and complex dynamical systems with delayed extremal feedback.

Declarations

Funding: This research has no funded.

Data availability: This manuscript has no associated data.

Ethical Conduct: Not applicable.

Conflicts of interest: The authors declare that there is no conflict of interest.

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