ISSN: 1694-9242
Journal of Osh University "Differential Equations", 2026, Volume 1, Issue 2, P. 1–10.
https://doi.org/10.52754/16949242_2026_2_1

UDC 517.927.4

Inverse Boundary Problem for a Fractional Order Integro-Differential Equations in a Locally Convex Space

Tursun K. Yuldashev*
Tashkent State Transport University, Temiryolchilar Street, 1, 100167, Tashkent, Ozbekistan
V. I. Romanovskiy Institute of Mathematics of Ozbekistan Academy of Science, Tashkent, Ozbekistan
Osh State University, Lenin street, 331, 723500, Osh, Kyrgyzstan
E-mail: tursun.k.yuldashev@gmail.com, https://orcid.org/0000-0002-9346-5362

Jaxongir T. Jonqobilov
Almalyk State Technical Institute, Ulugbek Street, 45, 110100, Almalyk, Tashkent, Ozbekistan
E-mail: jonqobilovjaxongir70@gmail.com, https://orcid.org/0009-0007-8056-9190

Muzaffar T. Sabirzhanov
Osh State University, Lenin Street, 331, 723500, Osh, Kyrgyzstan
E-mail: msabirjanov@oshsu.kg.com, https://orcid.org/0009-0008-2431-7033

Received: February 11, 2026; **First on-line published**: June 25, 2026

Abstract. *This paper investigates an inverse boundary problem for a nonlinear fractional-order integro-differential equation in a locally convex space. The considered model contains a nonlinear Volterra-type integral structure together with a fractional Gerasimov–Caputo operator, which reflects hereditary and memory effects arising in distributed parameter systems and nonlocal dynamical processes.*
Unlike the classical Banach-space framework, the analysis is developed in a locally convex setting generated by a family of seminorms. This approach makes it possible to study a significantly broader class of infinite-dimensional functional systems, including nonnormable spaces naturally appearing in generalized function theory, operator analysis, and evolution equations.
The inverse problem is reduced to a coupled nonlinear system consisting of two integral equations: one equation determines the main unknown function, while the second identifies the unknown boundary value. Since each equation contains both unknown quantities, the resulting system possesses a strongly nonlinear and nonlocal structure.
To establish solvability, the method of successive approximations is combined with a seminorm-based contraction principle in locally convex spaces. Existence and uniqueness of the solution pair are proved, and continuous dependence of the main function on the boundary data is established. The obtained results extend several classical solvability results for fractional integro-differential equations from Banach spaces to the wider class of locally convex spaces and provide a functional-analytic framework for studying inverse problems with memory effects in nonnormable settings.

Key words: *fractional order equation, locally convex space, equations with two unknown quantities, method of successive approximations, contraction mapping, existence, uniqueness, stability.*

Introduction

Fractional differential and integro-differential equations have become one of the principal mathematical tools for describing systems with memory, hereditary effects, anomalous diffusion, and nonlocal interactions. Such equations naturally arise in viscoelasticity, continuum mechanics, control theory, mathematical biology, heat transfer, porous media, and many other areas of applied science [1]–[22].

At the same time, inverse and boundary identification problems for fractional systems have attracted increasing attention due to their importance in parameter reconstruction, state estimation, and nonlocal observation problems. In many practical applications, unknown boundary values or hidden parameters must be reconstructed simultaneously with the state function itself. This leads to coupled nonlinear systems whose analysis becomes substantially more difficult in the presence of fractional operators and nonlinear integral terms.

Most existing works on fractional inverse problems are formulated in Banach spaces and rely on norm-based analytical methods. However, many naturally occurring functional spaces are not normable. Typical examples include Fréchet spaces, spaces of smooth functions, spaces of distributions, weak topologies of Banach spaces, and various inductive or projective limits. In such situations, locally convex spaces provide the natural analytical framework.

The transition from Banach spaces to locally convex spaces is mathematically nontrivial. Unlike normed spaces, the topology of a locally convex space is generated by a family of seminorms rather than by a single norm. Consequently, convergence, compactness, continuity, and contraction properties must be analyzed simultaneously with respect to all seminorms defining the topology.

Motivated by these considerations, the present paper studies an inverse boundary problem for a nonlinear fractional-order integro-differential equation involving the Gerasimov–Caputo operator in a locally convex space. The considered equation contains a nonlinear Volterra-type integral term under the action of nonlinear mappings, which leads to additional analytical difficulties caused by the interaction between nonlocality, nonlinearity, and fractional memory effects.

A distinctive feature of the paper is that the state function and the unknown boundary value are determined simultaneously from a coupled nonlinear system of integral equations. The resulting equations are strongly interconnected because each equation contains both unknown quantities.

The main contributions of the paper are as follows:

- reduction of the original inverse fractional boundary-value problem to an equivalent coupled nonlinear integral system;
- derivation of an explicit relation for the unknown boundary value;
- development of a seminorm-based analytical framework in locally convex spaces;
- proof of existence and uniqueness of the solution pair by combining successive approximations with contraction mapping techniques;
- establishment of continuous dependence of the solution on the boundary data.

The developed approach can be applied to a broad class of inverse problems for fractional differential and integro-differential equations in infinite-dimensional and nonnormable functional spaces.

1 Preliminary information and statement of the problem

Let E be a linear space over the field F of real numbers. A seminorm on E is a non-negative function $E : p \to R_+$ satisfying the homogeneity condition $p(\lambda x) = |\,\lambda\,| \, p(x)$ for all $\lambda \in F$ and the triangle inequality $p(x + y) \le p(x) + p(y)$ for all $x, y \in E$.

Definition 1. A sequence $\{x_n\} \in E$ is said to converge to an element $x \in E$ if for any $\varepsilon > 0$ and $p \in \mathscr{P}$ there exists a positive integer N such that $p(x_n - x) < \varepsilon$ for all $n \ge N$.

Definition 2. A sequence $\{x_n\} \in E$ is called a Cauchy sequence if for any $\varepsilon > 0$ and $p \in \mathscr{P}$ there exists a positive integer N such that $p(x_n - x_m) < \varepsilon$ for all $n, m \ge N$.

A locally convex space E is called complete if every Cauchy sequence is convergent.

So, let the space E be a complete locally convex space whose topology is defined by a family of seminorms

$$\mathscr{P} = \{p_\alpha : \, \alpha \in \mathscr{A}\} .$$

The operations of addition and multiplication by a number are continuous in any locally convex topology of the space E.

Let $\Omega = [0, T], \ 0 < T$, and let $C(\Omega, E)$ denote the space of all continuous functions $x : \Omega \to E$. For each seminorm $p_\alpha \in \mathscr{P}$ we define the induced seminorm on $C(\Omega, E)$:

$$\tilde{p}_\alpha(x) = \sup_{t\in\Omega} p_\alpha(x(t)).$$

Then $C(\Omega, E)$ is a locally convex space with respect to the family of seminorms $\{\tilde{p}_\alpha\}_{\alpha\in\mathscr{A}}$.

Let us consider the inverse problem for the following nonlinear integro-differential equations of fractional order

$$_C D^\gamma_{0t} x(t) = g(t, x(t)) \int\limits_0^t f\left(s, x(s)\right) ds, \quad t \in \Omega \tag{1.1}$$

with boundary condition

$$a\, x(0) + b\, x(T) = x_r \in E, \tag{1.2}$$

where $x(t)$ is the state function, x_r is the override value, $a, b \in \mathbb{R}, \ \ a \ne 0, \ b \ne 0, \ a \pm b \ne 0$, $u(t)$ is the control function, $_C D^\gamma_{0t}$ denotes the Gerasimov–Caputo fractional operator of order $0 < \gamma < 1$:

$$_C D^\gamma_{0t} x(t) = I^{1-\gamma}_{0t} x'(t) = \frac{1}{\Gamma(\gamma)} \int\limits_0^t \frac{x'(s) ds}{(t-s)^{1-\gamma}}, \ \ t \in \Omega,$$

and the Riemann–Liouville integral of order $0 < \gamma < 1$ is given by the expression

$$I^\gamma_{0t} x(t) = \frac{1}{\Gamma(\gamma)} \int\limits_0^t \frac{x(s) ds}{(t-s)^{1-\gamma}}, \ \ 0 < \gamma < 1,$$

$\Gamma(\gamma)$ is Gamma function: $\Gamma(z) = \int\limits_0^\infty t^{z-1} e^{-t} dt, \ \ \operatorname{Re} z > 0, \ g \in C(\Omega \times E, E), \ f \in C(\Omega \times E, E).$

Let us use the following internal condition

$$x(t_1) = x_r, \;\; 0 < t_1 < T. \tag{1.3}$$

Problem 1.1 (Statement of the problem)**.** It is required to find the function $x(t) \in C(\Omega, E)$ and redefinition value $x_r \in E$, that satisfy the equation (1.1), the boundary condition (1.2) and (1.3).

2 Reduction of the problem to the integral equations

Let $x(t) \in C(\Omega, E)$ be a solution to problem (1.1), (1.2). Apply the operator $I_{0\,t}^{\gamma}$ to equation (1.1):

$$I_{0\,t}^{\gamma}{}_C D_{0t}^{\gamma} x(t) = I_{0\,t}^{\gamma} g(t, x(t)) \int\limits_0^t f\left(s, x(s)\right) ds$$

and use the definition of the Gerasimov–Caputo fractional operator

$$I_{0\,t}^{\gamma}{}_C D_{0\,t}^{\gamma}\left(x(t)\right) = I_{0\,t}^{\gamma} I_{0\,t}^{1-\gamma} x'(t) = I_{0\,t}^{1} x'(t) = \int\limits_0^t x'(s) ds = x(t) - x(0),$$

$$I_{0t}^{\gamma} \eta_0\left(t, x(t)\right) = \frac{1}{\Gamma(\gamma)} \int\limits_0^t (t-s)^{\gamma-1} \eta_0\left(s, x(s)\right) ds.$$

Then, applying these two formulas to the last fractional integro-differential equation, we obtain:

$$x(t) = x(0) + \frac{1}{\Gamma(\gamma)} \int\limits_0^t (t-s)^{\gamma-1} g(s, x(s)) \int\limits_0^s f\left(\theta, x(\theta)\right)\, d\theta ds. \tag{2.1}$$

Here we use the boundary condition (1.2), then from the equality (2.1) we get that

$$x(0) = \frac{x_r}{a+b} - \frac{1}{\Gamma(\gamma)} \frac{b}{a+b} \int\limits_0^T (T-s)^{\gamma-1} g(s, x(s)) \int\limits_0^s f\left(\theta, x(\theta)\right) d\theta ds. \tag{2.2}$$

Now, we substitute the found value of the main function at the initial point into the integral equation (2.1):

$$x(t) = K_1(t; x, x_r) \equiv \frac{x_r}{a+b} + \int\limits_0^T V(t,s) g(s, x(s)) \int\limits_0^s f\left(\theta, x(\theta)\right) d\theta ds, \tag{2.3}$$

where

$$V(t,s) = \frac{1}{\Gamma(\gamma)} \begin{cases} (t-s)^{\gamma-1} - \frac{b}{a+b}(T-s)^{\gamma-1}, & t \le s \le T, \\ (t-s)^{\gamma-1}, \;\; 0 \le s \le t. \end{cases}$$

Equation (2.3) is a nonlinear Fredholm integral equation of the second kind and satisfies problem (1.1), (1.2).

We apply the intermediate condition (2.3) to the equation (1.3):

$$x(t_1) = \frac{x_r}{a+b} + \int_0^T V(t_1,s) g(s,x(s)) \int_0^s f(\theta, x(\theta))\, d\theta ds = x_r.$$

From here, by a simple transformation, we find that

$$x_r = K_2(t;x,x_r) \equiv \frac{a+b}{a+b-1} \int_0^T V(t_1,s) g(s,x(s)) \int_0^s f(\theta, x(\theta))\, d\theta ds. \tag{2.4}$$

The formula (2.4) is, like (2.3), a relation between unknown functions (x, x_r).

3 Key results

Let us consider the right-hand side of two operators (2.3), (2.4): $K_j(t;x,x_r,u),\ \ j=1,2$, as a system of equations.

Theorem 3.1. *Assume that the following conditions are satisfied:*
1). *The functions g, f, satisfy the inclusions:*

$$g \in C(\Omega \times E, E), f \in C(\Omega \times E, E),$$

and $p_\alpha(g) < \infty,\ \ p_\alpha(f) < \infty$;
2). *For each seminorm $p_\alpha \in \mathscr{P}$ there exist constants $0 < L_1, L_2$, such that the following inequalities hold:*

$$p_\alpha(g(t,x_1) - g(t,x_2)) \le L_1 p_\alpha(x_1 - x_2);$$

$$p_\alpha(f(t,x_1) - f(t,x_2)) \le L_2 p_\alpha(x_1 - x_2);$$

3). *For each seminorm p_α, the Schauder condition is satisfied*

$$\tilde{\rho}_{\alpha,\beta}(x,x_r) = \tilde{\rho}_\alpha(x) + \tilde{\rho} = \tilde{\chi}_1 + \tilde{\chi}_2 < 1,$$

where $\tilde{\chi}_1$, $\tilde{\chi}_2$ are determined from (3.11), (3.6), (3.8), *below.*

Then the system of two equations (2.3) *and* (2.4) *has a unique set of solutions $\{x(t) \in C(\Omega \times E),\ \ x_r \in E\}$. This pair of solutions can be found from the following iterative processes:*

$$\begin{cases} x^k(t) = K_1(t; x^{k-1}, x_r^{k-1}),\ \ k = 1,2,3,..., \\ x^0(t) = 0,\ \ t \in \Omega, \end{cases} \tag{3.1}$$

$$\begin{cases} x_r^k = K_2(t; x^{k-1}, x_r^{k-1}),\ \ k = 1,2,3,..., \\ x_r^0 = 0,\ \ t \in \Omega, \end{cases} \tag{3.2}$$

Proof. We fix $\alpha \in \mathscr{A}$. Then, taking into account the conditions of the theorem, for the first difference of approximations (3.1)–(3.2) the following estimate holds:

$$\tilde{p}_\alpha\left(x^{k+1}(t) - x^0(t)\right) \le \left|\frac{x_r^0}{a+b}\right| \sup_{t\in\Omega} \int_0^T |\, V(t,s)\,|\, p_\alpha\left(g(s,x^k(s))\right) \int_0^s p_\alpha\left(f\left(\theta, x^k(\theta)\right)\right) d\theta ds \le$$

$$\leq \left| \frac{x_r^0}{a+b} \right| \sup_{t\in\Omega} \int_0^T |\, V(t,s) \,| \, ds p_\alpha \left(g(t, x^k(t)) \right) p_\alpha \left(f\left(t, x^k(t) \right) \right) = \delta_1 < \infty. \tag{3.3}$$

Similarly, we obtain the following estimate

$$\left| x_r^{k+1} - x_r^0 \right| \leq \left| \frac{a+b}{a+b-1} \right| \int_0^T |\, V(t_1,s) \,| \; p_\alpha \left(g(s, x^k(s)) \right) \int_0^s p_\alpha \left(f\left(\theta, x^k(\theta) \right) \right) d\theta ds \leq$$

$$\leq b_0 \int_0^T |\, V(t_1,s) \,| \, ds \, p_\alpha \left(g(t, x^k(t)) \right) p_\alpha \left(f\left(t, x^k(t) \right) \right) = \delta_2 < \infty, \tag{3.4}$$

where $b_0 = \left| \dfrac{a+b}{a+b-1} \right|$.

Now for two pairs $\left(x^{k+1}(t), x_r^{k+1} \right)$ and $\left(x^k(t), x_r^k \right)$ from (3.1)–(3.2), using the Lipschitz conditions, we obtain

$$\tilde{p}_\alpha \left(x^{k+1}(t) - x^k(t) \right) \leq \left| \frac{x_r^k - x_r^{k-1}}{a+b} \right| + \sup_{t\in\Omega} \int_0^T |\, V(t,s) \,| \, p_\alpha \left(g(s, x^k(s)) \right) \times$$

$$\times \int_0^s L_2 p_\alpha \left(x^k(\theta) - x^{k-1}(\theta) \right) d\theta ds +$$

$$+ \sup_{t\in\Omega} \int_0^T |\, V(t,s) \,| \, L_1 p_\alpha \left(x^k(s) - x^{k-1}(s) \right) \int_0^s p_\alpha \left(f(\theta, x^{k-1}(\theta)) \right) d\theta ds \leq$$

$$\leq \left| \frac{x_r^k - x_r^{k-1}}{a+b} \right| + \sup_{t\in\Omega} \int_0^T |\, V(t,s) \,| \, ds \, p_\alpha \left(g(t, x^k(t)) \right) L_2 p_\alpha \left(x^k(t) - x^{k-1}(t) \right) +$$

$$+ \sup_{t\in\Omega} \int_0^T |\, V(t,s) \,| \, ds \, p_\alpha \left(f(t, x^{k-1}(t)(t)) \right) L_1 p_\alpha \left(x^k(t) - x^{k-1}(t) \right) \leq$$

$$\leq \left| \frac{x_r^k - x_r^{k-1}}{a+b} \right| + \chi_1 \tilde{p}_\alpha \left(x^k(t) - x^{k-1}(t) \right), \tag{3.5}$$

where

$$\chi_1 = \sup_{t\in\Omega} \int_0^T |\, V(t,s) \,| \, ds \left[L_2 p_\alpha \left(g(t, x^k(t)) \right) + L_1 p_\alpha \left(f(t, x^{k-1}(t)) \right) \right]. \tag{3.6}$$

Similarly, we obtain the estimate

$$\left| x_r^{k+1} - x_r^k \right| \leq \left| \frac{a+b}{a+b-1} \right| \int_0^T |\, V(t_1,s) \,| \; p_\alpha \left(g(s, x^k(s)) \right) \times$$

$$\times \int_0^s L_2 p_\alpha \left(x^k(\theta) - x^{k-1}(\theta) \right) d\theta ds +$$

$$+ \left| \frac{a+b}{a+b-1} \right| \int_0^T |\, V(t_1, s) \,|\, L_1 p_\alpha \left(x^k(s) - x^{k-1}(s) \right) \int_0^s p_\alpha \left(f(\theta, x^{k-1}(\theta)) \right) d\theta ds \leq$$

$$\leq \left| \frac{a+b}{a+b-1} \right| \int_0^T |\, V(t_1, s) \,|\, sds\, p_\alpha \left(g(t, x^k(t)) \right) L_2 p_\alpha \left(x^k(t) - x^{k-1}(t) \right) +$$

$$+ \left| \frac{a+b}{a+b-1} \right| \int_0^T |\, V(t_1, s) \,|\, sds\, p_\alpha \left(f(t, x^{k-1}(t)) \right) L_1 p_\alpha \left(x^k(t) - x^{k-1}(t) \right) \leq$$

$$\leq \chi_2 \tilde{p}_\alpha \left(x^k(t) - x^{k-1}(t) \right), \tag{3.7}$$

where

$$\chi_2 = b_0 \int_0^T |\, V(t_1, s) \,|\, sds \left[L_2 p_\alpha \left(g(t, x^k(t)) \right) + L_1 p_\alpha \left(f(t, x^{k-1}(t)) \right) \right]. \tag{3.8}$$

We rewrite the estimates (3.5) and (3.7) as follows:

$$\tilde{p}_\alpha \left(x^{k+1}(t) - x^k(t) \right) \leq$$

$$\leq \tilde{\chi}_1 \left[\tilde{p}_\alpha \left(x^k(t) - x^{k-1}(t) \right) + \left| x_r^k - x_r^{k-1} \right| \right], \tag{3.9}$$

$$\left| x_r^{k+1} - x_r^k \right| \leq \tilde{\chi}_2 \left[\tilde{p}_\alpha \left(x^k(t) - x^{k-1}(t) \right) + \left| x_r^k - x_r^{k-1} \right| \right], \tag{3.10}$$

where

$$\tilde{\chi}_1 = \max \left\{ |\, a+b \,|^{-1}, \chi_1 \right\}, \;\; \tilde{\chi}_2 = \chi_2. \tag{3.11}$$

Adding (3.9) and (3.10) term by term, we obtain that

$$\tilde{p}_\alpha \left(x^{k+1}(t) - x^k(t), x_r^{k+1} - x_r^k \right) \leq$$

$$\leq \tilde{\rho}_\alpha \tilde{p}_\alpha \left(x^k(t) - x^{k-1}(t) \right) + \tilde{\rho} \left| x_r^k - x_r^{k-1} \right| \leq$$

$$\leq \tilde{\rho}_\alpha \tilde{p}_\alpha \left(x^k(t) - x^{k-1}(t), x_r^k - x_r^{k-1} \right), \tag{3.12}$$

where

$$\tilde{\rho}_\alpha = \tilde{\chi}_1, \;\; \tilde{\rho} = \tilde{\chi}_2.$$

Since $\tilde{\rho}_\alpha < 1$, it follows from the estimate (3.12) that the operators given by the right-hand sides of equations (2.3) and (2.4) are contracting with respect to each seminorm of $\tilde{\rho}_\alpha$. Since the space $C(\Omega \times E)$ is complete, estimates (3.3)–(3.4) and (3.5), (3.7) imply the existence and uniqueness of the functions $x(t) \in C(\Omega \times E)$ and $x_r \in E$. □

Theorem 3.2. *Let the conditions of Theorem 3.1 be satisfied. Then the state function $x(t) \in E$ depends continuously on the boundary condition x_r.*

Proof. Now we show the continuous dependence of the solution to problem (1.1)–(1.2) on the right-hand side of the boundary condition (1.2). Let $x_r, x_r' \in E$ be two different numbers and the functions $x_1(t)$, $x_2(t) \in C(\Omega \times E)$ be the corresponding solutions of problem (1.1)–(1.2). Then, similarly to the estimates (3.1) from equation (2.2) we obtain that

$$p_\alpha\left(x_1(t) - x_2(t)\right) \leq p_\alpha\left(x_r - x_r'\right) + \left(\tilde{\chi}_1 + \tilde{\chi}_2\right) p_\alpha\left(x_1(t) - x_2(t)\right). \tag{3.13}$$

Since $\chi_1 + \chi_2 \leq \rho(x, x_r) < 1$, it follows from the inequality (3.13) that

$$p_\alpha\left(x_1(t) - x_2(t)\right) \leq \frac{p_\alpha\left(x_r - x_r'\right)}{1 - \rho(x, x_r)}.$$

The right-hand side tends to zero as $x_r \to x_r'$, which proves the continuous dependence of the solution on the boundary value. Theorem 3.2 is proved. □

Conclusion

In this paper, an inverse boundary problem for a nonlinear fractional-order integro-differential equation has been investigated in the framework of locally convex spaces. The considered problem combines several analytically challenging features simultaneously: fractional memory effects generated by the Gerasimov–Caputo operator, nonlinear integral interactions, and the necessity of reconstructing an unknown boundary value together with the state function.

Unlike the classical Banach-space approach, the analysis has been developed in a topology generated by a family of seminorms. This framework allows the obtained results to remain applicable in a substantially wider class of infinite-dimensional functional spaces, including non-normable spaces frequently encountered in generalized analysis and operator theory.

The original inverse boundary-value problem was transformed into a coupled nonlinear system of integral equations. One equation determines the state function, while the second reconstructs the unknown boundary value. The strong coupling between these equations reflects the intrinsic nonlocal structure of the inverse problem.

Existence and uniqueness of the solution pair were established by combining the method of successive approximations with contraction arguments formulated in terms of seminorms. In addition, continuous dependence of the solution on the boundary data was proved, which provides a stability foundation for further theoretical and numerical investigations.

The proposed approach opens several possible directions for future research, including inverse problems with nonlocal conditions, systems with delay and memory, fractional evolution equations in Fréchet spaces, and optimal control problems in nonnormable functional settings. The developed methods may also be useful in the study of distributed parameter systems, hereditary processes, and inverse identification problems in mathematical physics.

Declarations

Funding: This research has no funded.

Data availability: This manuscript has no associated data.

Ethical Conduct: Not applicable.

Conflicts of interest: The authors declare that there is no conflict of interest.

References

[1] O. Kh. Abdullaev, O. Sh. Salmanov, T. K. Yuldashev, "Direct and inverse problems for a parabolic-hyperbolic equation involving Riemann–Liouville derivatives", *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. – Mathematics* **43** (1), 21–33 (2023).

[2] K. Diethelm, A. D. Freed, "On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity", *Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties* (F. Keil, W. Mackens, H. Voss, and J. Werther, Eds), Springer–Verlag, Heidelberg, 1999. 217–224.

[3] K. Diethelm, N. J. Ford, "Analysis of fractional differential equations", *J. Math. Anal. Appl.* **265**, 229–248 (2002).

[4] D. Delbosco, L. Rodino, "Existence and uniqueness for a nonlinear fractional differential equation", *J. Math. Anal. Appl.* **204**, 609–625 (1996).

[5] L. Gaul, P. Klein, S. Kempfle, "Damping description involving fractional operators", *Mech. Systems Signal Processing* No. 5, 81–88 (1991).

[6] W. G. Glockle, T. F. Nonnenmacher, "A fractional calculus approach of selfsimilar protein dynamics", *Biophys. J.* **68**, 46–53 (1995).

[7] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, Singapore, 2000.

[8] E. R. Kaufmann, E. Mboumi, "Positive solutions of a boundary value problem for a nonlinear fractional differential equation", *Electron. J. Qual. Theory Differ. Equ.* **2007** (3), 1–11 (2007).

[9] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies **204**. Elsevier Science B.V., Amsterdam, 2006.

[10] M. T. Kosmakova, T. K. Yuldashev, M. U. Sagdullaeva, "Inverse problem for a nonlinear pseudiparabolic differential equation with final condition and Gerasimov–Caputo operator", *Journal of Osh University "Differential Equations"* **1** (1), 1–14 (2026).

[11] V. Lakshmikantham, A. S. Vatsala, "Theory of fractional differential inequalities and applications", *Commun. Appl. Anal.* **11** (3-4), 395–402 (2007).

[12] F. Mainardi, "Some basic problems in continuum and statistical mechanics", *Fractals and Fractional Calculus in Continuum Mechanics.* (A. Carpinteri and F. Mainardi, Eds), 291–348, Springer-Verlag, Wien, 1997.

[13] M. K. Mamanov, G. A. Khasanov, Z. A. Madatova, "Optimal control problem for an impulsive systems of functional-differential equations with a nonlinear function under the sign of a first-order differential", *J. Osh University "Differential Equations"* **1** (1), 67–77 (2026).

[14] N. K. Ochilova, T. K. Yuldashev, "On a nonlocal boundary value problem for a degenerate parabolic-hyperbolic equation with fractional derivative", *Lobachevskii J. Math.* **43** (1), 229–236 (2022). https://doi.org/10.1134/S1995080222040175

[15] I. Podlubny, "Fractional differential equations", *Mathematics in Sciences and Engineering* **198**, Academic Press, San Diego, 1999.

[16] A. V. Pskhu, *Partial differential equation of fractional order*. Moscow, Nauka, 2000 (in Russian).

[17] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives. Theory and applications*, Gordon and Breach, Yverdon, 1993.

[18] T. K. Yuldashev, "Determining of coefficients and the classical solvability of a nonlocal boundary-value problem for the Benney—Luke integro-differential equation with degenerate kernel", *J. Mathematical Sciences* **254** (6), 793–807 (2021).

[19] T. K. Yuldashev, O. Kh. Abdullaev, "Inverse problems for the loaded parabolic-hyperbolic equation involves Riemann–Liouville operator", *Lobachevskii J. Math.* **44** (3), 1080–1090 (2023). https://doi.org/10.1134/S1995080223030034.

[20] T. K. Yuldashev, B. J. Kadirkulov, "On a boundary value problem for a mixed type fractional differential equations with parameters", *Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan* **47** (1), 112–123 (2021).

[21] T. K. Yuldashev, B. J. Kadirkulov, R. A. Bandaliyev, "On a mixed problem for Hilfer type fractional differential equation with degeneration", *Lobachevskii J. Mathematics* **43** (1), 263–274 (2022).

[22] T. K. Yuldashev, B. J. Kadirkulov, Kh. R. Mamedov, "An inverse problem for Hilfer type differential equation of higher order", *Bulletin of the Karaganda University. Mathematics series* **105** (1), 136–149 (2022).