Journal of Osh University "Differential Equations", 2026, Volume 1, Issue 1, P. 105–117.

UDC 517.927.4

On G. E. Hutchinson Population Model for Fractional Differential Equations with Maxima

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Received: January 19, 2026; **First on-line published**: February 9, 2026

Abstract. *The article is devoted to the G. E. Hutchinson population model for fractional functional-differential equations with maxima. The functional-differential equation with maxima and Gerasimov–Caputo fractional operator is considered under the initial-final conditions. The proposed functional-differential equation we consider as a mathematical model of population dynamics of a species. The generalized spectral Jakobi–Galerkin method is used. The unique solvability theorem is proved.*

Key words: *G. E. Hutchinson population model, initial-final problem, functional-differential equation with maxima, Gerasimov–Caputo fractional operator, generalized spectral Jakobi–Galerkin method, unique solvability.*

MSC 2020: 34A08, 34A34, 34B15, 34K37.

Introduction. Problem statement

Let $(t_0, T) \subset \mathbb{R}^+ \equiv [0, \infty)$ be a finite interval on the set of positive real numbers, and let $\alpha > 0$. The Riemann–Liouville $\alpha-$order fractional integral of a function $\eta(t)$ is defined as follows:

$$J^{\alpha}_{t_0 t}\eta(t) = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-s)^{\alpha-1}\eta(s)ds, \quad \alpha > 0, \;\; t \in (t_0, T),$$

where $\Gamma(\alpha)$ is Gamma function [1, p. 112].

For $n-1 < \alpha \leq n, \;\; n \in \mathbb{N}$, Riemann–Liouville $\alpha-$order fractional derivative of a function $\eta(t)$ is defined as follows [2, Vol. 1, p. 27]:

$$D^{\alpha}_{t_0 t}\eta(t) = \frac{d^n}{dt^n} J^{n-\alpha}_{t_0 t}\eta(t), \;\; t \in (t_0; T).$$

The Gerasimov–Caputo $\alpha-$order fractional derivative of a function $\eta(t)$ is defined [2, Vol. 1, p. 34] by

$$ {}_{C}D^{\alpha}_{t_0t}\eta(t) = J^{n-\alpha}_{t_0t}\eta^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)}\int\limits_{t_0}^{t}\frac{\eta^{(n)}(s)ds}{(t-s)^{\alpha-n+1}}, \ \ t \in (t_0; T). $$

By another point, differential equations are a tool for mathematical modeling of processes and studying their properties (see, for example, [3]–[6] and see also [7]–[15]). Mathematical modelling in ecology has a long history leading back to the famous works of Malthus [11], Quetelet [12] and Verhulst [13] on principals of demographic regulation. In the first part of the 20th century, the classical works of Lotka [10] and Volterra [14] on dynamics of interacting species radically transformed the entire idea of modelling in ecology. Their works for a short history of the implementation of mathematics in ecology (see, [15]), eventually resulted in the emergence of theoretical ecology as a new science. Note that the questions on the main mechanisms of populations regulation addressed by Lotka and Volterra are still in the focus of theoretical ecology! In partially, to describe the population dynamics of a species, it was proposed a mathematical model (see, [3], [5])

$$ \dot{x}(t) + rx(t) + rx^2(t) = 0, \quad t \geq 0, \tag{0.1} $$

where $x(t) = N(t) + K$, $N(t)$ is the population size of the species, K is the capacity of the habitat, r is the reproduction rate.

Equation (0.1) describes well the dynamics of population growth of protozoan microorganisms. In this regard, in 1948, G. E. Hutchinson proposed a model described by a differential equation with delay [7], [8]

$$ \dot{x}(t) + rx(t-h) + rx(t-h)x(t) = 0, \quad t \geq 0, \tag{0.2} $$

where $x(t) = N(t) + K$, h characterizes the average reproductive age of the species.

Hutchinson used equation (0.2) to describe changes in the size of a herd of cows on a pasture. However, equation (0.2) has been used by other scientists to describe other processes (see, for examples, [4]). Let us consider equation (0.2) with the initial condition $x(t) = \varphi(t)$, $t \in [-h, 0]$, $x(0) = \varphi_0$. For small $r > 0$ the equation (0.2) has a stability domain. According to this model, asymptotic stability of the equilibrium position occurs when $N(t) \equiv K$, $t \geq 0$, $0 < rh < \pi/2$. It should be noted that periodic fluctuations appear in the mathematical model for $rh > \pi/2$. Consequently, in Hutchinson's model, the reproductive rate r and the average reproductive age h of the species play a fundamental role.

In this paper we propose the following model to describe the population dynamics of the species

$$ {}_{C}D^{\alpha}_{0t}x(t) + r\max_{\tau\in[q_1t, q_2t]} x(\tau) + rx(t)\max_{\tau\in[q_1t, q_2t]} x(\tau) = 0, \quad t \geq 0, \tag{0.3} $$

where $x(t) = N(t)+K$, τ is a variable value that provides the maximum number of species on the segment $[q_1t, q_2t]$, q_2 characterizes the increasing size of the reproductive age, q_1 characterizes the decreasing size of the reproductive age of the species in different time, r is reproduction rate, ${}_{C}D^{\alpha}_{0t}$ is Gerasimov–Caputo fractional operator, For the quantities q_1, q_2 it is possible three cases: 1) $0 < q_1 < q_2 < 1$; 2) $0 < q_1 < 1$, $1 < q_2 < \infty$; 3) $1 < q_1 < q_2 < \infty$. We will study the second case. Ather cases will be studied similarly.

Theoretically, differential equations with maxima are considered, in partially, in [16]–[21].

The equation (0.3) we consider with initial-final (0.4) condition

$$x(0) = x_0, \quad x(t) = \varphi_0(t), \quad t \in (T, q_2 T). \tag{0.4}$$

The solution of the differential equation (0.3) with condition (0.4) is represented as follows

$$x(t) = \Im(t; x) \equiv x_0 -$$

$$-\frac{r}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\max_{\tau \in [q_1 s, q_2 s]} x(\tau) + x(s) \max_{\tau \in [q_1 s, q_2 s]} x(\tau) \right] ds, \ \ t \in (0, T). \tag{0.5}$$

1 The generalized spectral Jakobi-Galerkin method

Now, to the problem (0.3), (0.4), we apply the generalized spectral Jacobi–Galerkin method as a numerical realization of solution (0.5). This solution (0.5) is nonlinear Volterra type fractional integral equation. On the interval $(-1, 1)$ for the given numbers $\beta_1, \ \beta_2 > -1$ we consider standard Jacobi polynomial $J_n^{(\beta_1, \beta_2)}(\xi)$ of degree n with weight function $\Lambda^{(\beta_1, \beta_2)}(\xi) = (1-\xi)^{\beta_1}(1+\xi)^{\beta_2}$. For the standard Jacobi polynomial there is true the following relation

$$\int_{-1}^{1} J_n^{(\beta_1,\beta_2)}(\xi)\, J_m^{(\beta_1,\beta_2)}(\xi)\, \Lambda^{(\beta_1,\beta_2)}(\xi) d\xi = \gamma_m^{(\beta_1,\beta_2)} \delta_{m,n}, \tag{1.1}$$

where $\delta_{m,n}$ is the Kronecker function and

$$\gamma_m^{(\beta_1,\beta_2)}(\xi) = \begin{cases} \dfrac{2^{\beta_1+\beta_2+1}\,\Gamma\,(\beta_1+1)\,\Gamma\,(\beta_2+1)}{\Gamma\,(\beta_1+\beta_2+2)}, & m = 0, \\ \dfrac{2^{\,\beta_1+\beta_2+1}\,\Gamma\,(m+\beta_1+1)\,\Gamma\,(m+\beta_2+1)}{(2m+\beta_1+\beta_2+1)\;m!\,\Gamma\,(m+\beta_1+\beta_2+2)}, & m \geq 1. \end{cases}$$

From the (1.1) we note that the set of standard Jacobi polynomial $J_n^{(\beta_1,\beta_2)}(\xi)$ is a complete orthogonal system in the space $L^2_{\Lambda^{(\beta_1,\beta_2)}}(-1,1)$ with weight function $\Lambda^{(\beta_1,\beta_2)}(\xi)$. In particular, $J_0^{(\beta_1,\beta_2)}(\xi) = 1$.

The shifted Jacobi polynomial of variable t and degree n is defined by the following formula

$$\tilde{J}_n^{(\beta_1,\beta_2)}(t) = J_n^{(\beta_1,\beta_2)}\left(\frac{2t}{T} - 1\right), \ \ t \in (0, T)\,. \tag{1.2}$$

We note that the set of shifted Jacobi polynomial $\tilde{J}_n^{(\beta_1,\beta_2)}(t)$ is a complete orthogonal system with weight function $\Lambda_T^{(\beta_1,\beta_2)}(t) = (T-t)^{\beta_1} t^{\beta_2}$ in the space $L^2_{\Lambda^{(\beta_1,\beta_2)}}(0,T)$ and by the aid of (1.2) we have the analog of the (1.1)

$$\int_0^T \tilde{J}_n^{(\beta_1,\beta_2)}(t)\, \tilde{J}_m^{(\beta_1,\beta_2)}(t)\, \Lambda_T^{(\beta_1,\beta_2)}(t) dt = \left(\frac{T}{2}\right)^{\beta_1+\beta_2+1} \gamma_m^{(\beta_1,\beta_2)}(t)\, \delta_{m,n}. \tag{1.3}$$

For any integer $N \geq 0$ we denote by $\left\{\xi_j^{(\beta_1,\beta_2)}, \ \eta_j^{(\beta_1,\beta_2)}\right\}_{j=0}^{N}$ the nodes and the corresponding Christoffel numbers of the standard Jacobi–Gauss interpolation on the interval $(-1, 1)$. By the

$\tilde{P}_N(0,T)$ we denote the set of polynomials of degree at most N on the interval $(0,T)$ and by the $t_j^{(\beta_1,\beta_2)}$ we denote the shifted Jacobi–Gauss quadrature nodes on the interval $(0,T)$:

$$t_j^{(\beta_1,\beta_2)} = \frac{T}{2}\left(\xi_j^{(\beta_1,\beta_2)} + 1\right), \quad 0 \le j \le N.$$

By virtue of the property of the standard Jacobi–Gauss quadrature, implies that for any $\phi(t) \in \tilde{P}_{2N+1}(0,T)$ we have

$$\int_0^T \phi(t)\, \Lambda_T^{(\beta_1,\beta_2)}(t)dt = \left(\frac{T}{2}\right)^{\beta_1+\beta_2+1} \sum_{j=0}^{N} \phi\left(t_j^{(\beta_1,\beta_2)}\right) \eta_j^{(\beta_1,\beta_2)}. \tag{1.4}$$

By virtue of (1.10) from (1.3), we have for any $0 \le n+m \le 2N+1$:

$$\sum_{j=0}^{N} \tilde{J}_m^{(\beta_1,\beta_2)}\left(t_j^{(\beta_1,\beta_2)}\right) \tilde{J}_n^{(\beta_1,\beta_2)}\left(t_j^{(\beta_1,\beta_2)}\right) \eta_j^{(\beta_1,\beta_2)} = \gamma_m^{(\beta_1,\beta_2)} \delta_{m,n}.$$

By the aid of shifted Jacobi polynomial $\tilde{J}_n^{(\beta_1,\beta_2)}(t)$ we define the shifted generalized Jacobi function of degree n as (see, [22], [23])

$$P_n^{(\beta_1,\beta_2)}(t) = t^{\beta_2} \tilde{J}_n^{(\beta_1,\beta_2)}(t), \quad \beta_1,\, \beta_2 > -1, \;\; t \in (0,T)\,. \tag{1.5}$$

According to (1.3) and (1.11), we see that

$$\int_0^T P_n^{(\beta_1,\beta_2)}(t)\, P_m^{(\beta_1,\beta_2)}(t)\, \Lambda_T^{(\beta_1,-\beta_2)}(t)dt = \left(\frac{T}{2}\right)^{\beta_1+\beta_2+1} \gamma_m^{(\beta_1,\beta_2)} \delta_{m,n}.$$

By virtue of (1.10), for any $\varphi(t) = t^{2\beta_2}\phi(t)$ we have

$$\int_0^T \varphi(t)\, \Lambda_T^{(\beta_1,-\beta_2)}(t)dt = \left(\frac{T}{2}\right)^{\beta_1+\beta_2+1} \sum_{j=0}^{N} \left(t_j^{(\beta_1,\beta_2)}\right)^{-2\beta_2} \varphi\left(t_j^{(\beta_1,\beta_2)}\right) \eta_j^{(\beta_1,\beta_2)}. \tag{1.6}$$

By the aid of (1.12) we introduce the inner product and norm in $L^2_{\Lambda_T^{(\beta_1,-\beta_2)}}(0,T)$ as

$$\langle f, g\rangle_{\Lambda_T^{(\beta_1,-\beta_2)}} = \left(\frac{T}{2}\right)^{\beta_1+\beta_2+1} \sum_{j=0}^{N} \left(t_j^{(\beta_1,\beta_2)}\right)^{-2\beta_2} f\left(t_j^{(\beta_1,\beta_2)}\right) g\left(t_j^{(\beta_1,\beta_2)}\right) \eta_j^{(\beta_1,\beta_2)},$$

$$\| f \|_{N,\Lambda_T^{(\beta_1,-\beta_2)}} = \sqrt{\langle f, f\rangle_{\Lambda_T^{(\beta_1,-\beta_2)}}}.$$

We need to introduce finite $N-$dimensional fractional polynomial space [22], [23]

$$\tilde{F}_N^{(\beta_2)}(0,T) = \left\{ t^{\beta_2}\, \psi(t) :\, \psi(t) \in \tilde{P}_N^{(\beta_1,\beta_2)}(0,T) \right\} = \text{span} \left\{ P_n^{(\beta_1,\beta_2)}(t) :\, 0 \le n \le N \right\}.$$

Then we note that for any $\phi,\, \psi \in \tilde{F}_N^{(\beta_2)}(0,T)$ hold the equalities

$$(\phi, \psi)_{\Lambda_T^{(\beta_1,-\beta_2)}} = \langle \phi, \psi\rangle_{\Lambda_T^{(\beta_1,-\beta_2)}}\,,$$

$$\| \phi \|_{\Lambda_T^{(\beta_1,-\beta_2)}} = \| \phi \|_{N,\Lambda_T^{(\beta_1,-\beta_2)}} .$$

Now we make in integral equation (0.5) variable transformation $s = \dfrac{t\theta}{T}$, $\theta \in (0,T)$. Then the integral equation (0.5) we describe as

$$x(t) = x_0 - V\,x(t) =$$

$$= x_0 - \left(\frac{t}{T}\right)^{\alpha} r \int\limits_0^t (T-\theta)^{\alpha-1} \left[\max_{\tau\in\left[\frac{q_1 t\theta}{T},\frac{q_2 t\theta}{T}\right]} x(\tau) + x\left(\frac{t\theta}{T}\right) \max_{\tau\in\left[\frac{q_1 t\theta}{T},\frac{q_2 t\theta}{T}\right]} x(\tau) \right] d\theta. \tag{1.7}$$

For the fractional operator's order $0<\alpha<1$ we denote $\alpha-1=-\mu$, where $0<\mu=\text{const}$. Then for $U,\,\varphi \in \tilde{F}_N^{(1-\mu)}(0,T)$ we apply the generalized spectral Jacobi–Galerkin method to equation (1.13):

$$(U,\,\varphi)_{\Lambda_T}^{(-\mu,\mu-1)} = (x_0,\varphi)_{\Lambda_T}^{(-\mu,\mu-1)} - (V\,U,\varphi)_{\Lambda_T}^{(-\mu,\mu-1)} . \tag{1.8}$$

We set

$$U(t) = \sum_{m=0}^{N} x_m(t)\, P_m^{(-\mu,1-\mu)}(t), \quad \varphi(t) = P_n^{(-\mu,1-\mu)}(t), \quad 0 \le m,\ n \le N.$$

Then for (1.14) we have

$$\sum_{m=0}^{N} x_m(t) \left(P_m^{(-\mu,1-\mu)}(t), P_n^{(-\mu,1-\mu)}(t) \right)_{\Lambda_T}^{(-\mu,\mu-1)} =$$

$$= \left(x_0, P_n^{(-\mu,1-\mu)}(t) \right)_{\Lambda_T}^{(-\mu,\mu-1)} + \left(V\,U(t), P_n^{(-\mu,1-\mu)}(t) \right)_{\Lambda_T}^{(-\mu,\mu-1)} .$$

Hence, we come to nonlinear system

$$\bar{B}\,\bar{x} = \bar{G} - r\bar{\vartheta}\,(\bar{x}), \tag{1.9}$$

where we introduced designations:

$$\bar{x} = (x_0, x_1, \ldots, \bar{x}_N)^T , \quad B = (b_{n,m})_{0\le n,m \le N} ,$$

$$b_{n,m} = \left(P_m^{(-\mu,1-\mu)}(t), P_n^{(-\mu,1-\mu)}(t) \right)_{\Lambda_T}^{(-\mu,\mu-1)} = \left(\frac{T}{2}\right)^{2-2\mu} \gamma_m^{(-\mu,1-\mu)} \delta_{m,n},$$

$$\bar{G} = (G_0, G_1, \ldots, G_N)^T , \ \ G_n(t) = \left(x_0, P_n^{(-\mu,1-\mu)}(t) \right)_{\Lambda_T}^{(-\mu,\mu-1)} ,$$

$$\bar{\vartheta}\,(\bar{x}) = (\vartheta_0, \vartheta_1, \ldots, \vartheta_N)^T , \ \ \vartheta_n(x) = \left(V\,U(t), P_n^{(-\mu,1-\mu)}(t) \right)_{\Lambda_T}^{(-\mu,\mu-1)} ,$$

where by $(\vartheta_0, \vartheta_1, \ldots, \vartheta_N)^T$ we denoted the transposition of the matrix $(\vartheta_0, \vartheta_1, \ldots, \vartheta_N)$.

We use the following quadrature formula

$$\langle W, V \rangle_{\Lambda_T}^{(\beta_1,-\beta_2)} = \left(\frac{T}{2}\right)^{\beta_1+\beta_2+1} \sum_{j=0}^{N} \left(t_j^{(\beta_1,\beta_2)} \right)^{-2\beta_2} W\left(t_j^{(\beta_1,\beta_2)} \right) V\left(t_j^{(\beta_1,\beta_2)} \right) \eta_j^{(\beta_1,\beta_2)}$$

to obtain approximate formulas:

$$G_n(t) \approx \left\langle x_0, P_n^{(-\mu,1-\mu)}(t) \right\rangle_{\Lambda_T}^{(-\mu,\mu-1)} =$$

$$= \left(\frac{T}{2}\right)^{2-2\mu} \sum_{j=0}^{N} \left(t_j^{(-\mu,1-\mu)}\right)^{2\mu-2} G\left(t_j^{(-\mu,1-\mu)}\right) P_n^{(-\mu,1-\mu)}\left(t_j^{(-\mu,1-\mu)}\right) \eta_j^{(-\mu,1-\mu)},$$

$$\bar{\vartheta}\left(\bar{x}\right) \approx \frac{(T)^{2-2\mu}}{2^{2-2\mu}} \sum_{i,j=0}^{N} \left(\frac{t_i^{(-\mu,1-\mu)}}{T}\right)^{1-\mu} (T-\theta)^{-\mu} \left[\max_{\tau\in\left[q_1\frac{t_i^{(-\mu,1-\mu)}t_j^{(-\mu,0)}}{T}, q_2\frac{t_i^{(-\mu,1-\mu)}t_j^{(-\mu,0)}}{T}\right]} x(\tau) + \right.$$

$$\left. + x\left(\frac{t_i^{(-\mu,1-\mu)}t_j^{(-\mu,0)}}{T}\right) \max_{\tau\in\left[q_1\frac{t_i^{(-\mu,1-\mu)}t_j^{(-\mu,0)}}{T}, q_2\frac{t_i^{(-\mu,1-\mu)}t_j^{(-\mu,0)}}{T}\right]} x(\tau)\right] \times$$

$$\times P_n^{(-\mu,1-\mu)}\left(t_i^{(-\mu,1-\mu)}\right) \eta_i^{(-\mu,1-\mu)} \eta_j^{(-\mu,0)}.$$

In approximately solving the system (1.15) one can use the Newton iterative method.

If in the model (0.3) we take into account the heterogeneous habitat, migration factors associated with the amount of available food, and the influence of predators, then the mathematical model takes on a more complex form

$$_C D_{0t}^{\alpha} x(t) + r \max_{\tau\in[q_1 t, q_2 t]} x(\tau) + r x(t) \max_{\tau\in[q_1 t, q_2 t]} x(\tau) = f\big(t, x(t)\big), \quad t \geq 0, \tag{1.10}$$

where f is an external disturbing influence on the population dynamics of a species.

We will consider equation (1.10) with the following conditions

$$x(0) + x(T) = c_0, \quad x(t) = \varphi_0(t), \quad t \in (T, q_2 T). \tag{1.11}$$

We integrate the problem (1.10), (1.11) on the interval $(0, t)$:

$$x(t) = x(0) +$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[-r \max_{\tau\in[q_1 s, q_2 s]} x(\tau) - r x(s) \max_{\tau\in[q_1 s, q_2 s]} x(\tau) + f(s, x(s)) \right] ds. \tag{1.12}$$

Taking into account condition (1.11), from equation (1.12) we obtain

$$x(0) = \frac{c_0}{2} -$$

$$- \frac{1}{2\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left[-r \max_{\tau\in[q_1 s, q_2 s]} x(\tau) - r x(s) \max_{\tau\in[q_1 s, q_2 s]} x(\tau) + f(s, x(s)) \right] ds. \tag{1.13}$$

Substituting (1.13) into equation (1.12), we obtain the nonlinear Fredholm integral equation

$$x(t) = J(t; x(t)) \equiv \frac{c_0}{2} +$$

$$+\int_0^T K_0(t,s)\left[-r\max_{\tau\in[q_1s,q_2s]}x(\tau)-rx(s)\max_{\tau\in[q_1s,q_2s]}x(\tau)+f(s,x(s))\right]ds, \tag{1.14}$$

where

$$K_0(t,s)=\begin{cases}-\dfrac{1}{2\Gamma(\alpha)}(T-s)^{\alpha-1}, & t\le s\le T,\\ \dfrac{1}{2\Gamma(\alpha)}(t-s)^{\alpha-1}-\dfrac{1}{2\Gamma(\alpha)}(T-s)^{\alpha-1}, & 0\le s<t.\end{cases}$$

We use a Banach space $C([0,T],\mathbb{R})$, that contains a function $u(t)$, defined and continuous on a segment $[0,T]$, with the norm

$$\|u(t)\|_{C[0,T]}=\max_{0\le t\le T}|u(t)|.$$

Let us also consider the Banach space

$$BD([0,T],\mathbb{R})=\{u:[0,T]\to\mathbb{R};\ u(t)\in C([0,T],\mathbb{R})\},$$

in which the function exists and is bounded by the norm

$$\|u(t)\|_{BD[0,T]}=\|u(t)\|_{C[0,T]}+h\,\|\dot u(t)\|_{C[0,T]},$$

where $0<h=\text{const}$.

Lemma 1.1 ([18], [19]). *For the difference of two functions with maxima there holds the following estimate*

$$\left\|\max_{\tau\in[t-h,t]}x(\tau)-\max_{\tau\in[t-h,t]}y(\tau)\right\|_{C[0,T]}\le\|x(t)-y(t)\|_{C[0,T]}+h\,\|\dot x(t)-\dot y(t)\|_{C[0,T]}.$$

Theorem 1.1. *Let be fulfilled the following conditions:*
1). $M_f=\|f(t,x)\|_{C[0,T]}$, $0<M_f=\text{const}$;
2). $|f(t,x_1)-f(t,x_2)|\le L_f\,|x_1-x_2|$, $0<L_f=\text{const}$;
3). $\rho_i=\max\left\{\rho_{i1};h\rho_{i2}\right\}<1$, $i=1,2,\ldots,n-1$.

Then for the equation (1.14) *has a unique solution in the class* $BD([0,T],\mathbb{R})$*, which can be found by following iteration process:*

$$x_0(t)=\frac{c_0}{2},\quad x_n(t)=J(t;x_{n-1}(t)),\ n=1,2,3,\ldots \tag{1.15}$$

Proof. For the first approximation from the equation (1.14) and approximations (1.15) we have

$$\|x_1(t)\|\le\frac{|c_0|}{2}+$$

$$+\frac{1}{2}\int_0^T|K_0(t,s)|\left[|r|\left\|\max_{\tau\in[q_1s,q_2s]}x_0(\tau)\right\|+|r|\,\|x_0(s)\|\left\|\max_{\tau\in[q_1s,q_2s]}x_0(\tau)\right\|+\|f(s,x_0(s))\|\right]ds\le$$

$$\le\frac{|c_0|}{2}+\frac{1}{2}\left[|r|\max\left\{\|\varphi_0(t)\|,\frac{|c_0|}{2}\right\}+|r|\frac{|c_0|}{2}\max\left\{\|\varphi_0(t)\|,\frac{|c_0|}{2}\right\}+M_f\right]\int_0^T|K_0(t,s)|\,ds=$$

$$= \frac{|\, c_0 |}{2} + \rho_{01} < \infty, \tag{1.16}$$

where

$$\rho_{01} = \frac{1}{2}\left[|\, r\,| \max\left\{ \|\, \varphi_0(t)\,\|\,, \frac{|\, c_0|}{2} \right\} + |\, r\,| \frac{|\, c_0|}{2} \max\left\{ \|\, \varphi_0(t)\,\|\,, \frac{|\, c_0|}{2} \right\} + M_f \right] \int\limits_0^T |K_0(t,s)\,|\, ds.$$

Taking into account estimate (1.16) from approximations (1.15) we have estimates

$$\|\, x_1(t) - x_0(t)\,\| \le \frac{1}{2} \int\limits_0^T |K_0(t,s)\,| \times$$

$$\times \left[|\, r\,| \left\| \max_{\tau\in[q_1 s, q_2 s]} x_0(\tau) \right\| + |\, r\,|\, \|\, x_0(s)\,\| \left\| \max_{\tau\in[q_1 s, q_2 s]} x_0(\tau) \right\| + \|\, f(s, x_0(s))\,\| \right] ds = \rho_{01}. \tag{1.17}$$

Analogously, from (1.10) we have

$$\|\, \dot{x}_1(t) - \dot{x}_0(t)\,\| \le$$

$$\le |\, r\,| \left\| \max_{\tau\in[q_1 t, q_2 t]} x_0(\tau) \right\| + |\, r|\, \|\, x_0(t)\,\| \left\| \max_{\tau\in[q_1 t, q_2 t]} x_0(\tau) \right\| + \|\, f(t, x_0(t))\,\| \le$$

$$\le |\, r| \max\left\{ \|\, \varphi_0(t)\,\|\,, \frac{|\, c_0|}{2} \right\} + |\, r| \frac{|\, c_0|}{2} \max\left\{ \|\, \varphi_0(t)\,\|\,, \frac{|\, c_0|}{2} \right\} + M_f = \rho_{02}. \tag{1.18}$$

From (1.17) and (1.18) we obtain that there holds the estimate

$$\|\, x_1(t) - x_0(t)\,\|_{C[0,T]} \le \rho_{01} + h\,\rho_{02}.$$

For the second difference we have

$$\|\, x_2(t) - x_1(t)\,\|_{C[0,T]} \le \left[\frac{|\, r\,|}{2} \left\| \max_{\tau\in[q_1 t, q_2 t]} x_1(\tau) - \max_{\tau\in[q_1 t, q_2 t]} x_0(\tau) \right\| + \right.$$

$$+ \frac{|\, r|}{2} \|\, x_1(t)\,\| \left\| \max_{\tau\in[q_1 t, q_2 t]} x_1(\tau) - \max_{\tau\in[q_1 t, q_2 t]} x_0(\tau) \right\| +$$

$$\left. + \frac{|\, r|}{2} \|\, x_1(t) - x_0(t)\,\| \left\| \max_{\tau\in[q_1 t, q_2 t]} x_0(\tau) \right\| + \frac{1}{2} \|\, f(t, x_1(t)) - f(t, x_0(t))\,\| \right] \int\limits_0^T |K_0(t,s)\,|\, ds.$$

We apply the lemma to the last inequality:

$$\|\, x_2(t) - x_1(t)\,\|_{C[0,T]} \le \left[\frac{|\, r|}{2} \left(\|\, x_1(t) - x_0(t)\,\| + h\, \|\, \dot{x}_1(t) - \dot{x}_0(t)\,\| \right) + \right.$$

$$+ \frac{|\, r\,|}{2} \left(\frac{|\, c_0|}{2} + \rho_{01} \right) \left(\|\, x_1(t) - x_0(t)\,\| + h\, \|\, \dot{x}_1(t) - \dot{x}_0(t)\,\| \right) +$$

$$\left. + \frac{|\, r\,|}{2} \|\, x_1(t) - x_0(t)\,\| \max\left\{ \|\, \varphi_0(t)\,\|\,, \frac{|\, c_0|}{2} \right\} + \frac{1}{2} L_f \|\, x_1(t) - x_0(t)\,\| \right] \int\limits_0^T |K_0(t,s)\,|\, ds.$$

Hence, we obtain that there holds

$$\| x_2(t) - x_1(t) \|_{C[0,T]} \le \chi_{11} \| x_1(t) - x_0(t) \|_{C[0,T]} + \chi_{12} h \, \| \dot{x}_1(t) - \dot{x}_0(t) \|_{C[0,T]} , \tag{1.19}$$

where

$$\chi_{11} = \frac{1}{2} \left[| r | \left(1 + \left(\frac{| c_0 |}{2} + \rho_{01} \right) + \max \left\{ \| \varphi_0(t) \| , \frac{| c_0 |}{2} \right\} \right) + L_f \right] \int_0^T | K_0(t,s) | \, ds,$$

$$\chi_{12} = \frac{| r |}{2} \left(1 + \frac{| c_0 |}{2} + \rho_{01} \right) \int_0^T | K_0(t,s) | \, ds.$$

Since $\chi_{11} \ge \chi_{12}$, the estimate (1.19) we rewrite as

$$\| x_2(t) - x_1(t) \|_{C[0,T]} \le \rho_{11} \left(\| x_1(t) - x_0(t) \|_{C[0,T]} + h \, \| \dot{x}_1(t) - \dot{x}_0(t) \|_{C[0,T]} \right) \le$$

$$\le \rho_{11} \left(\rho_{01} + h \, \rho_{02} \right) , \tag{1.20}$$

where $\rho_{11} = \chi_{11}$.

Analogously we derive that

$$\| \dot{x}_2(t) - \dot{x}_1(t) \|_{C[0,T]} \le | r | \left(\| x_1(t) - x_0(t) \| + h \, \| \dot{x}_1(t) - \dot{x}_0(t) \| \right) +$$

$$+ | r | \left(\frac{| c_0 |}{2} + \rho_{01} \right) \left(\| x_1(t) - x_0(t) \| + h \, \| \dot{x}_1(t) - \dot{x}_0(t) \| \right) +$$

$$+ | r | \, \| x_1(t) - x_0(t) \| \max \left\{ \| \varphi_0(t) \| , \frac{| c_0 |}{2} \right\} + L_f \, \| x_1(t) - x_0(t) \| \le$$

$$\le \chi_{21} \| x_1(t) - x_0(t) \|_{C[0,T]} + \chi_{22} h \, \| \dot{x}_1(t) - \dot{x}_0(t) \|_{C[0,T]} ,$$

$$\chi_{21} = | r | \left(1 + \left(\frac{| c_0 |}{2} + \rho_{01} \right) + \max \left\{ \| \varphi_0(t) \| , \frac{| c_0 |}{2} \right\} \right) + L_f,$$

$$\chi_{22} = | r | \left(1 + \frac{| c_0 |}{2} + \rho_{01} \right) .$$

From last estimate we get

$$\| \dot{x}_2(t) - \dot{x}_1(t) \|_{C[0,T]} \le \rho_{12} \left(\| x_1(t) - x_0(t) \|_{C[0,T]} + h \, \| \dot{x}_1(t) - \dot{x}_0(t) \|_{C[0,T]} \right) \le$$

$$\le \rho_{12} \left(\rho_{01} + h \, \rho_{02} \right) , \tag{1.21}$$

where $\rho_{12} = \chi_{21}$.

From (1.20) and (1.21) we obtain that there hold the estimates

$$\| x_2(t) - x_1(t) \|_{BD[0,T]} \le \rho_1 \| x_1(t) - x_0(t) \|_{BD[0,T]} ,$$

where $\rho_1 = \max\{\rho_{11}; h\rho_{12}\} < 1$;

$$\| x_2(t) - x_1(t) \|_{C[0,T]} \le \rho_{11} \left(\rho_{01} + h \, \rho_{02} \right) + h \, \rho_{12} \left(\rho_{01} + h \, \rho_{02} \right) =$$

$$= (\rho_{11} + h\,\rho_{12})\,(\rho_{01} + h\,\rho_{02})\,. \tag{1.22}$$

We need in the following estimate

$$\|\,x_2(t)\,\|_{C[0,T]} \leq \frac{|\,c_0|}{2} +$$

$$+\frac{1}{4}\int\limits_0^T \left[|\,r\,|\left\|\max_{\tau\in[q_1 s, q_2 s]} x_1(\tau)\right\| + |\,r|\,\|\,x_1(s)\,\|\left\|\max_{\tau\in[q_1 s, q_2 s]} x_1(\tau)\right\| + \|\,f(s, x_1(s))\,\|\right] ds \leq$$

$$\leq \frac{|\,c_0|}{2} + \frac{1}{2}\left[|\,r\,|\left(\max\left\{\|\,\varphi_0(t)\,\|\,, \frac{|\,c_0|}{2}\right\} + \rho_{01}\right) +\right.$$

$$\left. + |\,r\,|\left(\frac{|\,c_0|}{2} + \rho_{01}\right)\left(\max\left\{\|\,\varphi_0(t)\,\|\,, \frac{|\,c_0|}{2}\right\} + \rho_{01}\right) + M_f\right]\int\limits_0^T |K_0(t,s)\,|\,ds = \frac{|\,c_0|}{2} +$$

$$+\frac{1}{2}\left[|\,r\,|\left(\max\left\{\|\,\varphi_0(t)\,\|\,, \frac{|\,c_0|}{2}\right\} + \rho_{01}\right)\left(1 + \frac{|\,c_0|}{2} + \rho_{01}\right) + M_f\right]\int\limits_0^T |\,K_0(t,s)\,|\,ds < \infty. \tag{1.23}$$

Taking into account (1.23), analogously to (1.21) and (1.22) we consider next difference

$$\|\,x_3(t) - x_2(t)\,\|_{C[0,T]} \leq \left[\frac{|\,r\,|}{2}\left\|\max_{\tau\in[q_1 t, q_2 t]} x_2(\tau) - \max_{\tau\in[q_1 t, q_2 t]} x_1(\tau)\right\| +\right.$$

$$+\frac{|r|}{2}\,\|\,x_2(t)\,\|\left\|\max_{\tau\in[q_1 t, q_2 t]} x_2(\tau) - \max_{\tau\in[q_1 t, q_2 t]} x_1(\tau)\right\| +$$

$$\left.+\frac{|r|}{2}\,\|\,x_2(t) - x_1(t)\,\|\left\|\max_{\tau\in[q_1 t, q_2 t]} x_1(\tau)\right\| + \frac{1}{2}\,\|\,f(t, x_2(t)) - f(t, x_1(t))\,\|\right]\int\limits_0^T |K_0(t,s)\,|\,ds \leq$$

$$\leq \rho_{21}\left(\|\,x_2(t) - x_1(t)\,\|_{C[0,T]} + h\,\|\,\dot{x}_2(t) - \dot{x}_1(t)\,\|_{C[0,T]}\right), \tag{1.24}$$

where we denoted

$$\rho_{21} = \frac{1}{2}\left[|\,r\,|\left(\|\,x_2(t)\,\|_{C[0,T]} + \left\|\max_{\tau\in[q_1 t, q_2 t]} x_1(\tau)\right\|_{C[0,T]}\right) + L_f\right]\int\limits_0^T |\,K_0(t,s)\,|\,ds.$$

Similarly, from (1.21) and (1.22) we obtain

$$\|\,\dot{x}_3(t) - \dot{x}_2(t)\,\|_{C[0,T]} \leq \rho_{22}\left(\|\,x_2(t) - x_1(t)\,\|_{C[0,T]} + h\,\|\,\dot{x}_2(t) - \dot{x}_1(t)\,\|_{C[0,T]}\right), \tag{1.25}$$

where

$$\rho_{22} = |\,r|\left(\|\,x_2(t)\,\|_{C[0,T]} + \left\|\max_{\tau\in[q_1 t, q_2 t]} x_1(\tau)\right\|_{C[0,T]}\right) + L_f.$$

From the estimates (1.24) and (1.25) we obtain that there hold the estimates

$$\|\,x_3(t) - x_2(t)\,\|_{BD[0,T]} \leq \rho_2\,\|\,x_2(t) - x_1(t)\,\|_{BD[0,T]}\,,$$

where $\rho_2 = \max\{\rho_{21}; h\rho_{22}\} < 1$;

$$\| x_3(t) - x_2(t) \|_{C[0,T]} \leq \rho_{21} (\rho_{11} + h \rho_{12}) (\rho_{01} + h \rho_{02}) + h \rho_{22} =$$

$$= (\rho_{21} + h \rho_{22}) (\rho_{11} + h \rho_{12}) (\rho_{01} + h \rho_{02}) . \tag{1.26}$$

Continuing the process (1.26), we obtain an estimate for arbitrary n:

$$\| x_n(t) - x_{n-1}(t) \|_{BD[0,T]} \leq \rho_{n-1} \| x_{n-1}(t) - x_{n-2}(t) \|_{BD[0,T]} ,$$

where $\rho_{n-1} = \max \left\{\rho_{(n-1)1}; h\rho_{(n-1)2}\right\} < 1$;

$$\| x_n(t) - x_{n-1}(t) \|_{C[0,T]} \leq$$

$$\leq \left(\rho_{(n-1)1} + h \rho_{(n-1)2}\right) \cdots (\rho_{21} + h \rho_{22}) (\rho_{11} + h \rho_{12}) (\rho_{01} + h \rho_{02}) , \tag{1.27}$$

where

$$\rho_{(n-1)1} = \frac{1}{2} \left[| r | \left(\| x_{n-1}(t) \| + \left\| \max_{\tau \in [q_1 t, q_2 t]} x_{n-2}(\tau) \right\| \right) + L_f \right] \int\limits_0^T | K_0(t,s) | \, ds,$$

$$\rho_{(n-1)2} = | r | \left(\| x_{n-1}(t) \| + \left\| \max_{\tau \in [q_1 t, q_2 t]} x_{n-2}(\tau) \right\| \right) + L_f .$$

From (1.28) we obtain the following estimate:

$$\| x_n(t) - x_{n-1}(t) \|_{C[0,T]} \leq \rho_0 \rho_1 \rho_2 \cdot \ \ldots \ \cdot \rho_{n-2} \rho_{n-1}, \tag{1.28}$$

where $\rho_i = \max \left\{\rho_{i1}; h\rho_{i2}\right\} < 1, \ \ i = 1, 2, \ ..., \ n-1.$

The quantities r, h, L_f we choose such that from (1.28) implies

$$\lim_{n \to \infty} \| x_n(t) - x_{n-1}(t) \|_{BD[0,T]} = 0.$$

Hence, we deduce that the right-hand side of the equation (1.14) is contracting operator and the problem (1.10), (1.11) has a unique solution in the space $BD\left([0,T], \mathbb{R}\right)$. □

Declarations

Funding: There is no research Funding.
Data availability: This manuscript has no associated data.
Ethical Conduct: Not applicable.
Conflicts of interest: The authors declare that there is no conflict of interest.

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