Journal of Osh University "Differential Equations", 2026, Volume 1, Issue 1, P. 93–104.

UDC 517.956.2

Boundary Value Problems with Mixed Dirichlet and Neumann Conditions for Three-Dimensional Degenerate Elliptic Equation

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Received: January 16, 2026; **First on-line published**: February 03, 2026

Abstract. This article investigates two problems with mixed Dirichlet and Neumann conditions for a three-dimensional degenerate elliptic equation. Fundamental solutions of the named equation are expressed through a triple Lauricella hypergeometric function and explicit solutions of the mixed problems in the first octant are written out through a double Appell hypergeometric function. The energy integral method is used to prove the uniqueness of the solutions to the problems under consideration. In the course of proving the existence of the problem solution, differentiation formulas, decomposition formulas, some adjacent relations formulas and the auto-transformation formula of hypergeometric functions are used. The Gauss–Ostrogradsky formula is used to express problem's solutions in an explicit forms.

Key words: *Appell and Lauricella hypergeometric functions, three-dimensional degenerate elliptic equation, PDE-systems of hypergeometric type, fundamental solutions, mixed problems with Dirichlet and Neumann conditions, energy-integral method.*

MSC 2020: 33C05, 33C65, 35A08, 35J25, 35J70.

Introduction

Special functions are used for solving many problems of mathematical physics. These include the Gauss hypergeometric series, Bessel and Hermite functions, double Appell functions, multiple Lauricella hypergeometric functions, etc. The Hermite functions are actively applied in algorithms and information systems that are used in medical diagnostics [1]. The Bessel functions are used in solving a number of problems of hydrodynamics, radiophysics, and thermal conductivity [2]. Some functions that are used in astronomy can be arranged in hypergeometric series [3]. Multidimensional hypergeometric functions are used in the superstrings theory [4]. The theory of boundary value problems for degenerate equations is one of the important directions of the modern theory of partial differential equations, which has many applications in aerodynamics and gas dynamics [5]–[7] and irrigation problems [8]. A new properties and applications of Lauricella functions also are found in [9]–[11].

In formulation of local and nonlocal boundary value problems and construction of their explicit solutions, the main role are played fundamental solutions of the considered equations. In particular, boundary value problems for the degenerate elliptic equations

$$y^m u_{xx} + u_{yy} = 0,\ \ m > 0, y > 0$$

were the subject of interest of many mathematicians, such as Tricomi, Gellerstedt, Holmgren, Frankl, Pulkin and others (for instance, see [12]).

The theory of boundary value problems for the equation for the two-dimensional degenerate equation

$$y^m u_{xx} + x^n u_{yy} = 0,\ \ m > 0,\ \ n > 0,\ \ x > 0,\ \ y > 0$$

was developed in the last quarter of the twentieth century by Salakhitdinov and Hasanov [13] and Amanov [14].

1 Problems statement

In this paper, we first investigate two mixed Dirichlet–Neumann problems to a following three-dimensional degenerate equation

$$E_{\alpha,\beta,\gamma}(u) \equiv y^m z^k u_{xx} + x^n z^k u_{yy} + x^n y^m u_{zz} = 0,\ \ m > 0,\ n > 0,\ k > 0 \tag{1.1}$$

in the octant $\Omega = \left\{(x, y, z) : x > 0,\ y > 0,\ z > 0\right\}$. Note, a solution to the classic Dirichlet problem in the first octant for equation (1.1) is constructed in the explicit form [15]. In our recent work, we also obtained a solution to the classical Neumann problem for the equation (1.1) in explicit form, which is now awaiting publication in a scientific journal.

We introduce the notation:

$$\begin{gathered}
J_1 = \left\{(x, y, z) :\ \ x = 0,\ \ 0 < y,\ z < \infty\right\},\\
J_2 = \left\{(x, y, z) :\ \ y = 0,\ \ 0 < x,\ z < \infty\right\},\\
J_3 = \left\{(x, y, z) :\ \ z = 0,\ \ 0 < x,\ y < \infty\right\};\\
q = \frac{n+2}{2},\ \ p = \frac{m+2}{2},\ \ l = \frac{k+2}{2};\\
R = \sqrt{\frac{1}{q^2}x^{2q} + \frac{1}{p^2}y^{2p} + \frac{1}{l^2}z^{2l}},\ \ x > 0,\ y > 0,\ z > 0.
\end{gathered} \tag{1.2}$$

Dirichlet–Neumann problem $D_3^2 N_3^1$. Find a regular solution $u \in C\left(\overline{\Omega}\right) \cap C^1\left(\Omega \cup J_1\right) \cap C^2(\Omega)$ of equation (1.1), satisfying the following conditions

$$u\left(x, y, z\right)\big|_{z=0} = \tau_1\left(x, y\right),\ \ 0 \le x, y < \infty, \tag{1.3}$$

$$u\left(x, y, z\right)\big|_{y=0} = \tau_2\left(x, z\right),\ \ 0 \le x, z < \infty, \tag{1.4}$$

$$\lim_{x\to 0} \frac{\partial u(x, y, z)}{\partial x} = \nu_3(y, z),\ \ 0 < y, z < \infty, \tag{1.5}$$

$$\lim_{R\to\infty} u\left(x, y, z\right) = 0, \tag{1.6}$$

where $\tau_1\left(x, y\right), \tau_2\left(x, z\right), \nu_3\left(y, z\right)$ are given continuous functions, and for sufficiently large values of R the following inequalities are satisfied

$$\tau_1\left(x, y\right) = \frac{\tilde{\tau}_1\left(x, y\right)}{\left(1 + \frac{1}{q^2}x^{2q} + \frac{1}{p^2}y^{2p}\right)^{\varepsilon_1}},\ \ \tilde{\tau}_1(x, y) \in C(0 \le x, y < \infty), \tag{1.7}$$

$$\tau_2(x,z)=\frac{\tilde{\tau}_2(x,z)}{\left(1+\dfrac{1}{q^2}x^{2q}+\dfrac{1}{l^2}z^{2l}\right)^{\varepsilon_2}},\quad \tilde{\tau}_2(x,z)\in C(0\le x,z<\infty),\tag{1.8}$$

$$\nu_3(y,z)=\frac{\tilde{\nu}_3(y,z)}{\left(1+\dfrac{1}{p^2}y^{2p}+\dfrac{1}{l^2}z^{2l}\right)^{1/2-\alpha+\varepsilon_3}},\quad \tilde{\nu}_3(y,z)\in C(0<y,z<\infty),\tag{1.9}$$

and $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$ are some positive numbers.

In addition, the functions $\tau_1(x,y)$ and $\tau_2(x,z)$ satisfy the matching conditions at the origin: $\tau_1(0,0)=\tau_2(0,0)$ and at the lateral edge of domain Ω: $\tau_1(x,0)=\tau_2(x,0),\ 0\le x<\infty$.

Neumann–Dirichlet problem $N_3^2D_3^1$. Find a regular solution $u\in C\left(\overline{\Omega}\right)\cap C^1\left(\Omega\cup J_2\cup J_3\right)\cap C^2(\Omega)$ of equation (1.1), satisfying the condition (1.6) and the following conditions

$$\lim_{z\to 0}\frac{\partial u(x,y,z)}{\partial z}=\nu_1(x,y),\quad 0<x,y<\infty,\tag{1.10}$$

$$\lim_{y\to 0}\frac{\partial u(x,y,z)}{\partial y}=\nu_2(x,z),\quad 0<x,z<\infty,\tag{1.11}$$

$$u(x,y,z)|_{x=0}=\tau_3(y,z),\quad 0\le y,z<\infty,\tag{1.12}$$

where $\nu_1(t,s),\ \nu_2(t,s),\ \tau_3(t,s)$ are given functions, and for sufficiently large values of R the following inequalities are satisfied

$$\nu_1(x,y)=\frac{\tilde{\nu}_1(x,y)}{\left(1+\dfrac{1}{q^2}x^{2q}+\dfrac{1}{p^2}y^{2p}\right)^{1/2-\gamma+\varepsilon_1}},\quad \tilde{\nu}_1(x,y)\in C(0<x,y<\infty),\tag{1.13}$$

$$\nu_2(x,z)=\frac{\tilde{\nu}_2(x,z)}{\left(1+\dfrac{1}{q^2}x^{2q}+\dfrac{1}{l^2}z^{2l}\right)^{1/2-\beta+\varepsilon_2}},\quad \tilde{\nu}_2(x,z)\in C(0<x,z<\infty),\tag{1.14}$$

$$\tau_3(y,z)=\frac{\tilde{\tau}_3(y,z)}{\left(1+\dfrac{1}{p^2}y^{2p}+\dfrac{1}{l^2}z^{2l}\right)^{\varepsilon_3}},\quad \tilde{\tau}_3(y,z)\in C(0\le y,z<\infty),\tag{1.15}$$

and $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$ are some positive numbers.

2 The uniqueness theorems

One can readily check the validity of the following relation

$$uE(v)-vE(u)=y^mz^k\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial x}\right)+x^nz^k\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial y}-v\frac{\partial u}{\partial y}\right)+x^ny^m\frac{\partial}{\partial z}\left(u\frac{\partial v}{\partial z}-v\frac{\partial u}{\partial z}\right).$$

We denote by D_R the finite part of the region D, bounded by the planes $x=0,\ y=0,\ z=0$ and the half-quarter part

$$S_R:=\left\{(x,y,z):\frac{1}{q^2}x^{2q}+\frac{1}{p^2}y^{2p}+\frac{1}{l^2}z^{2l}=R^2,\ \ q>1,\ p>1,\ l>1\right\}$$

of the higher-order ellipsoid.

Let $D_{R,\varepsilon}$ be a sub-domain of D_R at a distance $\varepsilon>0$ from its boundary and $\cos(N,x)dS=dydz$, $\cos(N,y)dS=dxdz$, $\cos(N,z)dS=dxdy$, N is the outer normal to $\partial D_{R,\varepsilon}$. Integrate both sides of above given equality on the domain D_R and use the classical formula of Gauss–Ostrogradsky:

$$\iiint\limits_{D_{R,\varepsilon}}\big(uE(v)-vE(u)\big)dxdydz=\iint\limits_{\partial D_{R,\varepsilon}}\left[y^mz^k\left(u\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial x}\right)\cos(N,x)+\right.$$

$$+x^n z^k \left(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y} \right) \cos(N,y) + x^n y^m \left(u\frac{\partial v}{\partial z} - v\frac{\partial u}{\partial z} \right) \cos(N,z) \bigg] dS.$$

Using the equality

$$uE_{\alpha,\beta,\gamma}(u) + y^m z^k u_x^2 + x^n z^k u_y^2 + x^n y^m u_z^2 = y^m z^k \frac{\partial}{\partial x}(uu_x) + x^n z^k \frac{\partial}{\partial y}(uu_y) + x^n y^m \frac{\partial}{\partial z}(uu_z),$$

we obtain

$$\iiint\limits_{D_{R,\varepsilon}} uE(u)dxdydz + \iiint\limits_{D_{R,\varepsilon}} \left[y^m z^k u_x^2 + x^n z^k u_y^2 + x^n y^m u_z^2 \right] dxdydz =$$

$$= \iiint\limits_{D_{R,\varepsilon}} \left[y^m z^k \frac{\partial}{\partial x}(uu_x) + x^n z^k \frac{\partial}{\partial y}(uu_y) + x^n y^m \frac{\partial}{\partial z}(uu_z) \right] dxdydz.$$

Applying again the formula of Gauss–Ostrogradskii to this equality and letting $\varepsilon \to 0$, we get

$$\iiint\limits_{D} \left(y^m z^k u_x^2 + x^n z^k u_y^2 + x^n y^m u_z^2 \right) dxdydz = \iint\limits_{S_1} x^n y^m \tau_1(x,y)\nu_1(x,y)dxdy$$

$$+ \iint\limits_{S_2} x^n z^k \tau_2(x,z)\nu_2(x,z)dxdz + \iint\limits_{S_3} y^m z^k \tau_3(y,z)\nu_3(y,z)dydz + \iint\limits_{S_R} uC[u]dS_R, \qquad (2.1)$$

where $C[u] = y^m z^k u_x \cos(N,x) + x^n z^k u_y \cos(N,y) + x^n y^m u_z \cos(N,z)$.

Theorem 2.1. *The Dirichlet–Neumann problem $D_3^2 N_3^1$ with conditions* (1.3)–(1.6) *for equation* (1.1) *can have at most one solution.*

Proof. To prove the uniqueness of the solution, as usual, we suppose that the problem has two v, w solutions. Denoting $u = v - w$ we have that u satisfies homogeneous Dirichlet–Neumann problem $D_3^2 N_3^1$ ($\tau_1 = 0$, $\tau_2 = 0$, $\nu_3 = 0$, $\lim\limits_{R\to\infty} u = 0$). Further we have to prove that the homogeneous problem has only trivial solution. In this case from (2.1) one can easily get

$$\iiint\limits_{D} \left(y^m z^k u_x^2 + x^n z^k u_y^2 + x^n y^m u_z^2 \right) dxdydz = 0.$$

Hence, it follows that $u_x = u_y = u_z = 0$, which implies that u is a constant function. Considering the conditions (1.3) and (1.4), we conclude that $u(x,y,z) \equiv 0$ in $\overline{D}$. □

Theorem 2.2. *The Neumann–Dirichlet problem $N_3^2 D_3^1$ with conditions* (1.6), (1.10)–(1.12) *for equation* (1.1) *can have at most one solution.*

Proof. The Theorem 2.2 is proved similarly with the Theorem 2.1. □

3 Existence of a solution to Dirichlet-Neumann problem $D_3^2 N_3^1$

Consider a function

$$u(x,y,z) = \int\limits_0^\infty \int\limits_0^\infty t^n s^m \tau_1(t,s) \left. \frac{\partial}{\partial \zeta} q_{22}(x,y,z;t,s,\zeta) \right|_{\zeta=0} dtds +$$

$$+\int\limits_0^\infty\int\limits_0^\infty t^n s^k \tau_2(t,s)\,\frac{\partial}{\partial\zeta} q_{22}(x,y,z;t,\eta,s)\Big|_{\eta=0} dtds-\int\limits_0^\infty\int\limits_0^\infty t^m s^k \nu_3(t,s)\, q_{22}(x,y,z;\xi,t,s)\big|_{\xi=0} dtds, \quad (3.1)$$

where $q_{22}(x,y,z;\xi,\eta,\zeta)$ is a fundamental solution defined in [15]:

$$q_{22}(x,y,z;\xi,\eta,\zeta) = \frac{k_{22}yz\eta\zeta}{r^{5+2\alpha-2\beta-2\gamma}} F_A^{(3)}\left[\begin{matrix}\frac{5}{2}+\alpha-\beta-\gamma,\alpha,1-\beta,1-\gamma; \\ 2\alpha,2-2\beta,2-2\gamma;\end{matrix}\ \rho,\sigma,\theta\right], \quad (3.2)$$

$$\alpha=\frac{n}{2(n+2)},\ \ \beta=\frac{m}{2(m+2)},\ \ \gamma=\frac{k}{2(k+2)};$$

$$\rho=-\frac{4x^q\xi^q}{q^2r^2},\ \sigma=-\frac{4y^p\eta^p}{p^2r^2},\ \ \theta=-\frac{4z^l\zeta^l}{l^2r^2},\ \ r^2=\frac{1}{q^2}\left(x^q-\xi^q\right)^2+\frac{1}{p^2}\left(y^p-\eta^p\right)^2+\frac{1}{l^2}\left(z^l-\zeta^l\right)^2; \quad (3.3)$$

the numbers q, p and l are defined in (1.2);

$$k_{22}=q^{-2\alpha}p^{2\beta-2}l^{2\gamma-2}\frac{\Gamma(\alpha)\Gamma(1-\beta)\Gamma(1-\gamma)\Gamma(4+2\alpha-2\beta-2\gamma)}{2\pi\Gamma(2\alpha)\Gamma(2-2\beta)\Gamma(2-2\gamma)\Gamma(2+\alpha-\beta-\gamma)}. \quad (3.4)$$

It is obvious that

$$0<2\alpha<1,\ \ 0<2\beta<1,\ \ 0<2\gamma<1;\quad q>1,\ p>1,\ l>1.$$

Here $F_A^{(3)}$ denotes a famous Lauricella hypergeometric function [16]:

$$F_A^{(3)}\left[\begin{matrix}a,b_1,b_2,b_3; \\ c_1,c_2,c_3;\end{matrix}\ x,y,z\right]=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\ \ |x|+|y|+|z|<1,$$

where $(\lambda)_n$ is a Pochhammer symbol: $(\lambda)_n=\lambda(\lambda+1)...(\lambda+n-1),\ \ n=1,2,...;\ (\lambda)_0=1$.

Applying the differentiation formula [17, p. 19, Eq. (20)]

$$\frac{\partial^{m+n+p}}{\partial x^m\partial y^n\partial z^p}F_A^{(3)}\left[\begin{matrix}a,b_1,b_2,b_3; \\ c_1,c_2,c_3;\end{matrix}\ x,y,z\right]=\frac{(a)_{m+n+p}(b_1)_m(b_2)_n(b_3)_p}{(c_1)_m(c_2)_n(c_3)_p}\times$$

$$\times F_A^{(3)}\left[\begin{matrix}a+m+n+p,\ b_1+m,b_2+n,b_3+p; \\ c_1+m,c_2+n,c_3+p;\end{matrix}\ x,y,z\right],$$

from (3.1) we get the following function:

$$u(x,y,z)=u_1(x,y,z)+u_2(x,y,z)+u_3(x,y,z), \quad (3.5)$$

where

$$u_1(x,y,z)=k_{22}yz\int\limits_0^\infty\int\limits_0^\infty\frac{\tau_1(t,s)\,t^n s^{m+1}}{r_1^{5+2\alpha-2\beta-2\gamma}}F_2\left[\begin{matrix}\frac{5}{2}+\alpha-\beta-\gamma,\ \alpha,1-\beta; \\ 2\alpha,\ 2-2\beta;\end{matrix}\ -\frac{4x^qt^q}{q^2r_1^2},-\frac{4y^ps^p}{p^2r_1^2}\right]dtds, \quad (3.6)$$

$$u_2(x,y,z)=k_{22}yz\int\limits_0^\infty\int\limits_0^\infty\frac{\tau_2(t,s)\,t^n s^{k+1}}{r_2^{5+2\alpha-2\beta-2\gamma}}F_2\left[\begin{matrix}\frac{5}{2}+\alpha-\beta-\gamma,\ \alpha,1-\gamma; \\ 2\alpha,\ 2-2\gamma;\end{matrix}\ -\frac{4x^qt^q}{q^2r_2^2},-\frac{4z^ls^l}{l^2r_2^2}\right]dtds, \quad (3.7)$$

$$u_3(x,y,z)=$$

$$=-k_{22}yz\int\limits_0^\infty\int\limits_0^\infty\frac{\nu_3(t,s)\,t^{m+1}s^{k+1}}{r_3^{5+2\alpha-2\beta-2\gamma}}F_2\left[\begin{matrix}\frac{5}{2}+\alpha-\beta-\gamma,\ 1-\beta,1-\gamma; \\ 2-2\beta,\ 2-2\gamma;\end{matrix}\ -\frac{4y^pt^p}{p^2r_3^2},-\frac{4z^ls^l}{l^2r_3^2}\right]dtds, \quad (3.8)$$

$$r_1^2 = \frac{1}{q^2}\left(x^q - t^q\right)^2 + \frac{1}{p^2}\left(y^p - s^p\right)^2 + \frac{1}{l^2} z^{2l}, \tag{3.9}$$

$$r_2^2 = \frac{1}{q^2}\left(x^q - t^q\right)^2 + \frac{1}{p^2} y^{2p} + \frac{1}{l^2}\left(z^l - s^l\right)^2, \tag{3.10}$$

$$r_3^2 = \frac{1}{q^2} x^{2q} + \frac{1}{p^2}\left(y^p - t^p\right)^2 + \frac{1}{l^2}\left(z^l - s^l\right)^2. \tag{3.11}$$

Here F_2 is Appell hypergeometric function defined as follows [18]:

$$F_2\left[\begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\left(b_1\right)_m\left(b_2\right)_n}{\left(c_1\right)_m\left(c_2\right)_n} \frac{x^m}{m!}\frac{y^n}{n!}, \quad |x| + |y| < 1,$$

which satisfies the following system of partial differential equations [19, p. 234, Eq. 5.9(10)]:

$$\begin{cases} x(1-x)\dfrac{\partial^2 F_2}{\partial x^2} - xy\dfrac{\partial^2 F_2}{\partial x \partial y} + \left[c_1 - (a + b_1 + 1)\, x\right]\dfrac{\partial F_2}{\partial x} - b_1 y \dfrac{\partial F_2}{\partial y} - ab_1 F_2 = 0, \\ y(1-y)\dfrac{\partial^2 F_2}{\partial y^2} - xy\dfrac{\partial^2 F_2}{\partial x \partial y} - b_2 x\dfrac{\partial F_2}{\partial x} + \left[c_2 - (a + b_2 + 1)\, y\right]\dfrac{\partial F_2}{\partial y} - ab_2 F_2 = 0. \end{cases} \tag{3.12}$$

Lemma 3.1. *If the function* $\tau_1(x, y)$ *can be represented as* (1.7)*, then the function* $u_1(x, y, z)$ *defined in* (3.6) *is a regular solution of the equation* (1.1) *in the domain* Ω *satisfying the conditions* (1.6) *and*

$$u_1(x, y, 0) = \tau_1(x, y), \quad u_1(x, 0, z) = 0, \quad u_1(0, y, z) = 0. \tag{3.13}$$

Proof. First let us prove that the function (3.6) satisfies the degenerate elliptic equation (1.1). For this purpose, we consider the auxiliary function

$$W(x, y, z; t, s) = \frac{yz}{r_1^{5+2\alpha-2\beta-2\gamma}} \omega(\vartheta, \varsigma), \tag{3.14}$$

where

$$\omega(\vartheta, \varsigma) := F_2\left[\begin{matrix} \frac{5}{2} + \alpha - \beta - \gamma, \alpha, 1-\beta; \\ 2\alpha,\ 2-2\beta; \end{matrix} \vartheta, \varsigma\right], \quad \vartheta = -\frac{4x^q t^q}{q^2 r_1^2}, \quad \varsigma = -\frac{4y^p s^p}{p^2 r_1^2}.$$

We calculate the necessary derivatives of the auxiliary function W with respect to the variables x, y, z and substitute them into the degenerate elliptic equation (1.1), we obtain the following system of hypergeometric equations

$$\begin{cases} \vartheta(1-\vartheta)\omega_{\vartheta\vartheta} - \vartheta\varsigma\omega_{\vartheta\varsigma} + \left[2\alpha - \left(\frac{7}{2} + 2\alpha - \beta - \gamma\right)\vartheta\right]\omega_\vartheta - \alpha\left(\frac{5}{2} + \alpha - \beta - \gamma\right)\omega = 0, \\ \varsigma(1-\varsigma)\omega_{\varsigma\varsigma} - \vartheta\varsigma\omega_{\vartheta\varsigma} + \left[2 - 2\beta - \left(\frac{9}{2} + \alpha - 2\beta - \gamma\right)\varsigma\right]\omega_\varsigma - (1-\beta)\left(\frac{5}{2} + \alpha - \beta - \gamma\right)\omega = 0. \end{cases}$$

Comparing the last system of equations with the system of equations (3.12) for the Appell function F_2, we can conclude that the function (3.14) is a solution of the corresponding degenerate elliptic equation. Consequently, the function $u_1(x, y, z)$ defined by (3.6) satisfies the degenerate elliptic equation (1.1).

Now we prove that the function $u_1(x, y, z)$ satisfies the boundary conditions (3.13). Indeed, applying a famous expansion formula [20]

$$F_2\left[\begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix} x, y\right] = \sum_{j=0}^{\infty} \frac{(a)_j (b_1)_j (b_2)_j}{j!(c_1)_j (c_2)_j} x^j y^j F\left(\begin{matrix} a+j,\ b_1+j; \\ c_1+j, ; \end{matrix} x\right) F\left(\begin{matrix} a+j,\ b_2+j; \\ c_2+j; \end{matrix} y\right)$$

to the Appell function F_2 in the double integral (3.6), we get

$$u_1(x, y, z) = k_{22} yz \sum_{j=0}^{\infty} \frac{(\delta_2)_j (\alpha)_j (1-\beta)_j}{j!(2\alpha)_j (2-2\beta)_j} \int_0^\infty \int_0^\infty \frac{\tau_1(t, s)\, t^n s^{m+1}}{r_1^{5+2\alpha-2\beta-2\gamma}} \times$$

$$\times\vartheta^j\varsigma^j F\left(\begin{array}{c}\frac{5}{2}+\alpha-\beta-\gamma+j,\, \alpha+j;\\ 2\alpha+j;\end{array}\vartheta\right) F\left(\begin{array}{c}\frac{5}{2}+\alpha-\beta-\gamma+j,\, 1-\beta+j;\\ 2-2\beta+j;\end{array}\varsigma\right) dtds, \tag{3.15}$$

where $F(a,b;c;z)$ is a famous Gaussian hypergeometric function defined as follows [19, p.56, Eq. 2.1(2)]

$$F\left(a,b;c;z\right)\equiv F\left(\begin{array}{c}a,b;\\ c;\end{array} z\right)=\sum_{m=0}^{\infty}\frac{(a)_m\,(b)_m}{(c)_m}\frac{z^m}{m!},\ \ |z|<1.$$

Now we apply the well-known transformation formula [19, p.105, Eq. 2.9(4)]

$$F(a,b;c;z)=(1-z)^{-b}F\left(c-a,b;c;\frac{z}{z-1}\right)$$

to each Gauss hypergeometric function in (3.15), as a result of which we obtain

$$u_1\left(x,y,z\right)=k_{22}yz\sum_{j=0}^{\infty}\frac{\left(\delta_2\right)_j(\alpha)_j(1-\beta)_j}{j!(2\alpha)_j(2-2\beta)_j}\int\limits_0^{\infty}\int\limits_0^{\infty}\frac{\tau_1\left(t,s\right)t^n s^{m+1}}{r_1^{2\delta_2}}(1-\vartheta)^{-\alpha}(1-\varsigma)^{\beta-1}\times$$

$$\times\left(\frac{\vartheta}{\vartheta-1}\right)^j\left(\frac{\varsigma}{\varsigma-1}\right)^j F\left(\begin{array}{c}2\alpha-\delta_2,\, \alpha+j;\\ 2\alpha+j;\end{array}\frac{\vartheta}{\vartheta-1}\right) F\left(\begin{array}{c}2-2\beta-\delta_2,\, 1-\beta+j;\\ 2-2\beta+j;\end{array}\frac{\varsigma}{\varsigma-1}\right) dtds. \tag{3.16}$$

Introducing in the integrand in (3.16) instead of t and s new variables

$$\mu=\frac{l\left(t^q-x^q\right)}{qz^l},\quad \nu=\frac{l\left(s^p-y^p\right)}{pz^l},$$

and knowing the equalities

$$t=\left(x^q+\mu qz^l/l\right)^{1/q},\ \ s=\left(y^p+\nu pz^l/l\right)^{1/p},\ \ r_1^2=\frac{z^{2l}}{l^2}\left(1+\mu^2+\nu^2\right),$$

$$dt=\frac{1}{l}z^l\left(x^q+\mu qz^l/l\right)^{(1-q)/q}d\mu,\ \ ds=\frac{1}{l}z^l\left(y^p+\nu pz^l/l\right)^{(1-p)/p}d\nu,$$

$$1-\vartheta=\frac{q^2z^{2l}\left(1+\mu^2+\nu^2\right)+4l^2x^q\left(x^q+\mu qz^l/l\right)}{q^2z^{2l}\left(1+\mu^2+\nu^2\right)}\equiv\frac{r_1^2+4x^q\left(x^q+\mu qz^l/l\right)/q^2}{r_1^2},$$

$$1-\varsigma=\frac{p^2z^{2l}\left(1+\mu^2+\nu^2\right)+4l^2y^p\left(y^p+\nu pz^l/l\right)}{p^2z^{2l}\left(1+\mu^2+\nu^2\right)}\equiv\frac{r_1^2+4y^p\left(y^p+\nu pz^l/l\right)/p^2}{r_1^2},$$

we obtain

$$u_1(x,y,z)=\frac{k_{22}y}{l^{2\gamma-1}}\sum_{j=0}^{\infty}\frac{\left(\delta_2\right)_j(\alpha)_j(1-\beta)_j}{j!(2\alpha)_j(2-2\beta)_j}\int\limits_{-\frac{lx^q}{qz^l}}^{\infty}\int\limits_{-\frac{ly^p}{pz^l}}^{\infty}\frac{\tau_1\left[\left(x^q+\mu qz^l/l\right)^{1/q},\ \left(y^p+\nu pz^l/l\right)^{1/p}\right]}{\left(1+\mu^2+\nu^2\right)^{(3-2\gamma)/2}}\times$$

$$\times\frac{\left(x^q+\frac{\mu q}{l}z^l\right)^{2\alpha}\left(y^p+\frac{\nu p}{l}z^l\right)\left(\frac{\vartheta}{\vartheta-1}\right)^j\left(\frac{\varsigma}{\varsigma-1}\right)^j}{\left[\frac{z^{2l}}{l^2}\left(1+\mu^2+\nu^2\right)+\frac{4x^q}{q^2}\left(x^q+\frac{\mu q}{l}z^l\right)\right]^{\alpha}\left[\frac{z^{2l}}{l^2}\left(1+\mu^2+\nu^2\right)+\frac{4y^p}{p^2}\left(y^p+\frac{\nu p}{l}z^l\right)\right]^{1-\beta}}\times$$

$$\times F\left(\begin{array}{c}-\frac{5}{2}+\alpha+\beta+\gamma,\, \alpha+j;\\ 2\alpha+j,;\end{array}\frac{\vartheta}{\vartheta-1}\right) F\left(\begin{array}{c}-\frac{1}{2}-\alpha-\beta+\gamma,\, 1-\beta+j;\\ 2-2\beta+j;\end{array}\frac{\varsigma}{\varsigma-1}\right) d\mu d\nu. \tag{3.17}$$

Now on the right side of the equality (3.17) we pass to the limit at $z \to 0$. Taking into account obvious equalities

$$\lim_{z\to 0}\frac{\vartheta}{\vartheta-1}\equiv\lim_{z\to 0}\frac{4l^2x^q\left(x^q+\mu qz^l/l\right)}{q^2z^{2l}\left(1+\mu^2+\nu^2\right)+4l^2x^q\left(x^q+\mu qz^l/l\right)}=1,$$

$$\lim_{z\to 0}\frac{\varsigma}{\varsigma-1}\equiv\lim_{z\to 0}\frac{4l^2y^p\left(y^p+\nu qz^l/l\right)}{p^2z^{2l}\left(1+\mu^2+\nu^2\right)+4l^2y^p\left(y^p+\nu qz^l/l\right)}=1,$$

using the summation formula [19, p. 61, Eq. 2.1(14)]

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ \ \mathrm{Re}(c)>Re(a+b),\ \ c\neq 0,-1,-2,\dots,$$

considering the formula for calculating the double improper integral [21, p. 633, Eq. 4.623]

$$\int\limits_0^\infty\int\limits_0^\infty\varphi\left(a^2x^2+b^2y^2\right)dxdy=\frac{\pi}{4ab}\int\limits_0^\infty\varphi(x)dx$$

and Legendre's duplication formula [19, p. 5, Eq. 1.2(15)]

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

by virtue of the expression (3.4) of the coefficient k_{22}, we obtain

$$\lim_{z\to 0}u_1\left(x,y,z\right)=\tau_1\left(x,y\right). \tag{3.18}$$

Using the similar transformations, we have

$$\lim_{x\to 0}u_1\left(x,y,z\right)=0,\ \ \lim_{y\to 0}u_1\left(x,y,z\right)=0. \tag{3.19}$$

Therefore, based on equalities (3.18) and (3.19) we conclude that the function $u_1(x,y,z)$, defined by (3.6), satisfies conditions (3.13).

Let us show that if given function τ_1 has representation (1.7), then the function $u_1(x,y,z)$ defined in (3.6) tends to zero at infinity. Using the transformation formula for Appell function F_2 [19, p. 240, Eq. 5.11(8)]

$$F_2\left[\begin{matrix}a,b_1,b_2;\\ c_1,c_2;\end{matrix}\ x,y\right]=(1-x-y)^{-a}F_2\left[\begin{matrix}a,c_1-b_1,c_2-b_2;\\ c_1,c_2;\end{matrix}\ \frac{x}{x+y-1},\frac{y}{x+y-1}\right],$$

we write the function (3.6) in the form

$$u_1\left(x,y,z\right)=k_{22}yz\int\limits_0^\infty\int\limits_0^\infty\frac{\tau_1\left(t,s\right)t^{n+1}s^{m+1}}{\rho^{5+2\alpha-2\beta-2\gamma}}F_2\left[\begin{matrix}\frac{5}{2}+\alpha-\beta-\gamma,\alpha,1-\beta;\\ 2\alpha,\ 2-2\beta;\end{matrix}\ \frac{4x^qt^q}{q^2\rho^2},\frac{4y^ps^p}{p^2\rho^2}\right]dtds, \tag{3.20}$$

where

$$\rho^2=\frac{1}{q^2}\left(x^q+t^q\right)^2+\frac{1}{p^2}\left(y^p+s^p\right)^2+\frac{1}{l^2}z^{2l}.$$

It is easy to see that in (3.20) the following inequality holds

$$\frac{4x^qt^q}{q^2\rho^2}+\frac{4y^ps^p}{p^2\rho^2}<1,\quad x>0,\ \ y>0,\ \ z>0,\ \ t>0,\ \ s>0.$$

Let us prove that when the point (x, y, z) tends to infinity, i.e. when $R \to \infty$, the function (3.20) tends to zero. It known from the theory of Appell functions [18], that, if $|x| + |y| < 1$, then for any values of the numerical parameters the Appell hypergeometric function F_2 is bounded:

$$| F_2 (a, b_1, b_2; c_1, c_2; x, y) | \leq C_1, \quad |x| + |y| < 1.$$

Next, applying the representation (1.7) for given function $\tau_1(x, y)$, we obtain

$$|u_1| \leq C_2 yz \int_0^\infty \int_0^\infty \frac{t^{n+1} s^{m+1} dt ds}{\left(1 + \frac{1}{q^2} t^{2q} + \frac{1}{p^2} s^{2p}\right)^{\varepsilon_1} \left[\frac{1}{q^2} (x^q + t^q)^2 + \frac{1}{p^2} (y^p + s^p)^2 + \frac{1}{l^2} z^{2l}\right]^{5/2+\alpha-\beta-\gamma}}. \quad (3.21)$$

Substituting t and s for

$$\mu = \frac{1}{qR} t^q, \quad \nu = \frac{1}{pR} s^p$$

in the last double improper integral (3.21), we get

$$|u_1| \leq \frac{qpC_3}{R^{2\varepsilon_1 + 2\alpha - 2\beta - 2\gamma - 2}} \cdot \frac{x}{R} \cdot \frac{y}{R} \cdot \frac{z}{R} \cdot K(x, y; R), \quad (3.22)$$

where $\varepsilon_1 > 1 - \alpha + \beta + \gamma$ (see condition in (1.7)) and

$$K(x, y; R) = \int_0^\infty \int_0^\infty \frac{\mu \nu d\mu d\nu}{\left(\mu^2 + \nu^2 + \frac{1}{R^2}\right)^{\varepsilon_1} \left(1 + \mu^2 + \nu^2 + \frac{2x^q}{qR} + \frac{2y^p}{pR}\right)^{5/2+\alpha-\beta-\gamma}}. \quad (3.23)$$

It is easy to show that the double improper integral on the right-hand side (3.23) is bounded as $R \to \infty$. Indeed, using the formula [22]

$$\underbrace{\int_0^{+\infty} \cdots \int_0^{+\infty}}_{n} \frac{x_1^{p_1 - 1} \ldots x_n^{p_n - 1} dx_1 \ldots dx_n}{\left[(r_1 x_1)^{q_1} + \ldots + (r_n x_n)^{q_n}\right]^t \left[1 + (r_1 x_1)^{q_1} + \ldots + (r_n x_n)^{q_n}\right]^s} =$$

$$= \frac{\Gamma(p_1/q_1) \ldots \Gamma(p_n/q_n) \Gamma(P - t) \Gamma(s + t - P)}{q_1 q_2 \ldots q_n r_1^{p_1 q_1} \ldots r_n^{p_n q_n} \Gamma(P) \Gamma(s)}, \quad P := \frac{p_1}{q_1} + \ldots + \frac{p_n}{q_n},$$

where p_k, q_k, r_k and s are positive numbers $(k = \overline{1, n})$, $0 < P - t < s$, and passing in (3.23) to the limit as $R \to \infty$, we obtain

$$\lim_{R \to \infty} K(x, y; R) = \frac{\Gamma(2 - \varepsilon_1) \Gamma(1/2 + \alpha - \beta - \gamma + \varepsilon_1)}{4\Gamma(5/2 + \alpha - \beta - \gamma)}, \quad 1 - \alpha + \beta + \gamma < \varepsilon_1 < 2. \quad (3.24)$$

Thus, by virtue of (3.22) and (3.24) the following estimate is valid:

$$|u_1| \leq \frac{C_4}{R^{2(\varepsilon_1 + \alpha - \beta - \gamma - 1)}}, \; 1 - \alpha + \beta + \gamma < \varepsilon_1 < 2, \; R \to \infty. \quad (3.25)$$

Considering (3.25), we conclude that the function (3.6) vanishes at infinity. Lemma 3.1 is proved. □

Remark 3.1. Repeating the arguments given in Lemma 3.1, one can prove two lemmas concerning the functions $u_2(x, y, z)$ and $u_3(x, y, z)$ defined by the equalities (3.7) and (3.8), respectively. Thus, if the representations (1.8) and (1.9) are valid for the given functions $\tau_2(x, z)$ and $\nu_3(y, z)$, then each of the functions $u_2(x, y, z)$ and $u_3(x, y, z)$ is a solution to the degenerate elliptic equation (1.1) that vanishes at infinity and satisfies the set of conditions

$$u_2(x, y, 0) = 0, \quad u_2(x, 0, z) = \tau_2(x, z), \quad u_2(0, y, z) = 0,$$

$$\left.\frac{\partial u_3(x,y,z)}{\partial z}\right|_{z=0}=0,\quad \left.\frac{\partial u_3(x,y,z)}{\partial y}\right|_{y=0}=0,\quad \left.\frac{\partial u_3(x,y,z)}{\partial x}\right|_{x=0}=\nu_3(y,z),$$

respectively.

Theorem 3.1. *If given functions* $\tau_1(x,y)$, $\tau_2(x,z)$ *and* $\nu_3(y,z)$ *have the representations* (1.7), (1.8) *and* (1.9), *respectively, then the function* $u(x,y,z)$ *defined in* (3.5) *is a regular solution of the equation* (1.1) *in the domain* Ω *satisfying the conditions* (1.3)– (1.6).

Proof of Theorem 3.1 follows from the Lemma 3.1 and Remark 3.1.

4 Existence of a solution to Dirichlet-Neumann problem $N_3^2 D_3^1$

Consider a function

$$u(x,y,z)=-\int_0^\infty\int_0^\infty t^n s^m \nu_1(t,s)\, q_1(x,y,z;t,s,\zeta)|_{\zeta=0}\, dtds$$

$$-\int_0^\infty\int_0^\infty t^n s^k \nu_2(t,s)\, q_1(x,y,z;t,\eta,s)|_{\eta=0}\, dtds+\int_0^\infty\int_0^\infty t^m s^k \tau_3(t,s)\left.\frac{\partial}{\partial\xi}q_1(x,y,z;\xi,t,s)\right|_{\xi=0} dtds, \quad (4.1)$$

where $q_1(x,y,z;\xi,\eta,\zeta)$ is a fundamental solution defined in [15]:

$$q_1(x,y,z;\xi,\eta,\zeta)=\frac{k_1 x\xi}{r^{3-2\alpha+2\beta+2\gamma}}F_A^{(3)}\left[\begin{matrix}\frac{3}{2}-\alpha+\beta+\gamma,1-\alpha,\beta,\gamma; \\ 2-2\alpha,2\beta,2\gamma;\end{matrix}\ \rho,\sigma,\theta\right], \quad (4.2)$$

ρ, σ and θ are defined in (3.3);

$$k_1=q^{2\alpha-2}p^{-2\beta}l^{-2\gamma}\frac{\Gamma(1-\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(2-2\alpha+2\beta+2\gamma)}{2\pi\Gamma(2-2\alpha)\Gamma(2\beta)\Gamma(2\gamma)\Gamma(1-\alpha+\beta+\gamma)}.$$

By applying the reasoning from the previous section to the expression (4.1), one can to construct a solution of the Neumann–Dirichlet problem $N_3^2 D_3^1$ in the explicit form:

$$u(x,y,z)=-k_1 x\int_0^\infty\int_0^\infty \frac{\nu_1(t,s)\,t^{n+1}s^m}{r_1^{3-2\alpha+2\beta+2\gamma}}F_2\left[\begin{matrix}\frac{3}{2}-\alpha+\beta+\gamma,\ 1-\alpha,\beta; \\ 2-2\alpha,\ 2\beta;\end{matrix}\ -\frac{4x^q t^q}{q^2 r_1^2},-\frac{4y^p s^p}{p^2 r_1^2}\right]dtds-$$

$$-k_1 x\int_0^\infty\int_0^\infty \frac{\nu_2(t,s)\,t^{n+1}s^k}{r_2^{3-2\alpha+2\beta+2\gamma}}F_2\left[\begin{matrix}\frac{3}{2}-\alpha+\beta+\gamma,\ 1-\alpha,\gamma; \\ 2-2\alpha,\ 2\gamma;\end{matrix}\ -\frac{4x^q t^q}{q^2 r_2^2},-\frac{4z^l s^l}{l^2 r_2^2}\right]dtds+$$

$$+k_1 x\int_0^\infty\int_0^\infty \frac{\tau_3(t,s)\,t^m s^k}{r_3^{3-2\alpha+2\beta+2\gamma}}F_2\left[\begin{matrix}\frac{3}{2}-\alpha+\beta+\gamma,\beta,\gamma; \\ 2\beta,2\gamma;\end{matrix}\ -\frac{4y^p t^p}{p^2 r_3^2},-\frac{4z^l s^l}{l^2 r_3^2}\right]dtds, \quad (4.3)$$

where r_1, r_2 and r_3 are defined in (3.9), (3.10) and (3.11), respectively.

Theorem 4.1. *If given functions* $\nu_1(x,y)$, $\nu_2(x,z)$ *and* $\tau_3(y,z)$ *have the representations* (1.13), (1.14) *and* (1.15), *respectively, then the function* $u(x,y,z)$ *defined in* (4.3) *is a regular solution of the equation* (1.1) *in the domain* Ω *satisfying the conditions* (1.6) *and* (1.10)–(1.12).

Proof of Theorem 4.1 is carried out similarly to the proof of Theorem 3.1.

Conclusion

Thus, in this work and in [15], the main boundary value problems in an infinite domain (in the first octant) are solved. Note that using the fundamental solutions defined in (3.2) and (4.2), one can find a simple and double layer potentials, volume potentials, and also Green's functions associated with the equation (1.1), which are used in solving boundary value problems in the finite domains.

In recent paper [23] all fundamental solutions of the multidimensional degenerate elliptic equation

$$\sum_{k=1}^{n} \prod_{j=1, j\neq k}^{n} \left[x_j^{m_j+1}\right] \left(x_k \frac{\partial^2 u}{\partial x_k^2} + p_k \frac{\partial u}{\partial x_k} \right) = 0,\ 0 < p_k < 1,\ m_k > -2p_k$$

in the domain $R_+^n = \left\{ (x_1, ..., x_n) : x_1 > 0, ..., x_n > 0 \right\}$ are constructed in explicit forms which are expressed through the multiple Lauricella hypergeometric function $F_A^{(n)}$. Using the method used in [23], one can construct fundamental solutions of the equation

$$\sum_{k=1}^{n} \prod_{j=1,\, j\neq k}^{n} \left[x_j^{m_j}\right] \frac{\partial^2 u}{\partial x_k^2} = 0,\ \ m_j > 0,\ x_j > 0,\ j = \overline{1, n},$$

which is a multidimensional analogue of the equation (1.1) discussed in this paper.

Data availability This manuscript has no associated data.

Ethical Conduct Not applicable.

Conflicts of interest The authors declare that there is no conflict of interest.

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