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A Boundary Value Problem with the Generalized M. Saigo Operator for a Mixed-Type Equation with the Hilfer Fractional Derivative

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Abstract. This work studies the existence of a solution to a nonlocal boundary value problem involving the generalized M. Saigo operator for a diffusion equation with the Hilfer derivative and a degenerate hyperbolic equation in a bounded domain. The problem is reduced to an ordinary fractional differential equation with the Riemann–Liouville operator.

Key words: *Mixed type equation, a boundary value problem, Hilfer operator, M. Saigo operator, Bitsadze–Samarskii type conditions.*

MSC 2020: 35R10, 35R12.

Introduction. Problem statement

Fractional-order differential equations have attracted increasing interest due to their ability to capture memory and hereditary effects inherent in many physical and engineering systems. In this context, mixed-type and degenerate equations with fractional derivatives play an important role in the mathematical modeling of diffusion–wave processes in heterogeneous and complex media.

Fractional partial differential equations commonly arise in models describing phenomena in media with fractal or irregular structures [1]. Their analysis often requires nonstandard methods, particularly in the presence of nonlocal boundary conditions and fractional integro-differential operators. For instance, a mixed-type equation with a Riemann–Liouville fractional derivative subject to a nonlocal boundary condition was studied in [2], highlighting the significance of fractional operators in applied problems.

The unique solvability of boundary value problems for fractional diffusion equations coupled with degenerate hyperbolic equations has been investigated in several works. In particular, [3] established unique solvability in an unbounded domain with singular coefficients. Boundary value problems with integral conjugation conditions for mixed-type equations with the Hilfer derivative were studied in [4] using methods of integer- and fractional-order differential equations and integral equation techniques. Furthermore, linear mixed optimal control problems for fractional pseudoparabolic equations with degeneration were analyzed in [5], where necessary optimality conditions were derived via the Fourier series method and solved using successive approximations.

Consider the following differential equation

$$\begin{cases} ku_{xx} - D_{0+}^{\gamma,\delta}u = 0, \ \ y > 0, \\ y^{2m}u_{xx} + yu_{yy} + \alpha u_y = 0, \ \ y < 0 \end{cases} \tag{0.1}$$

in a mixed domain $D = D^+ \cup D^- \cup I$. Here k is positive constant, m is natural number, $\frac{1-2m}{2} < \alpha < 1$, $D^+ = \{(x, y) : 0 < x < 1, 0 < y < 1\}$, D^- is a bounded domain of the half-plane $y < 0$, bounded by the characteristics of equation (0.1)

$$OC : x - \frac{2}{2m+1}(-y)^{\frac{2m+1}{2}} = 0, \ \ BC : x + \frac{2}{2m+1}(-y)^{\frac{2m+1}{2}} = 1$$

starting from the points $O(0, 0)$ and $B(1, 0)$, and by the segment OB of the line $y = 0$, $I = \{(x, y) : 0 < x < 1, \ y = 0\}$,

$$\left(D_{0+}^{\gamma,\delta}f\right)(y) = \left(I_{0+}^{\delta(1-\gamma)}\frac{d}{dy}I_{0+}^{(1-\delta)(1-\gamma)}f\right)(y)$$

is the Hilfer fractional derivative of order $\gamma\,(0 < \gamma \le 1)$ and of type $\delta\,(0 \le \delta \le 1)$ [6], where

$$(I_{0+}^{\gamma}f)(y) = \frac{1}{\Gamma(\gamma)}\int_0^y \frac{f(z)dz}{(y-z)^{1-\gamma}}$$

is the Riemann–Liouville fractional integral of order $\gamma(\mathrm{Re}\ (\gamma) > 0)$ [7].

$I_{0+}^{\gamma,\delta,\eta}$ is the generalized fractional integro-differential operator with the Gauss hypergeometric function $F(a, b, c; z)$ introduced by M. A. Saigo [8] (see, [9] pp. 326–327) and having the form for real γ, δ, η and $x > 0$

$$(I_{0+}^{\gamma,\delta,\eta}f)(x) = \begin{cases} \dfrac{x^{-\gamma-\delta}}{\Gamma(\gamma)}\displaystyle\int_0^x (x-t)^{\gamma-1}F\left(\gamma+\delta, -\eta, \gamma; 1-\frac{t}{x}\right)f(t)dt, \ (\gamma > 0), \\ \dfrac{d^n}{dx^n}(I_{0+}^{\gamma+n,\delta-n,\eta-n}f)(x), (\gamma \le 0, n = [-\gamma]+1), \end{cases} \tag{0.2}$$

where $[-\gamma]$ is the integer part of $-\gamma$.

In particular,

$$(I_{0+}^{\gamma,-\gamma,\eta}f)(x) = (I_{0+}^{\gamma}f)(x), \ (I_{0+}^{-\gamma,\gamma,\eta}f)(x) = (D_{0+}^{\gamma}f)(x), \tag{0.3}$$

$$(I_{0+}^{0,0,\eta}f)(x) = f(x), \tag{0.4}$$

where $(I_{0+}^{\gamma}f)(x)$ and $(D_{0+}^{\gamma}f)(x)$ are the Riemann–Liouville fractional integration and differentiation operators of order $\gamma > 0$ (see, [9] pp.42–44).

Problem. Find a solution $u(x,y)$ of equation (0.1) in the domain D satisfying the following conditions:

$$u(0,y)=0, \quad u(1,y)=0, \quad 0\le y\le 1, \tag{0.5}$$

$$A_1(I^{-\beta,0,2\beta-1}u[\Theta(t)])x+A_2u(x,0)=g(x), \quad 0\le x\le 1, \tag{0.6}$$

as well as the conjugation conditions

$$\lim_{y\to 0+} y^{(1-\gamma)(1-\delta)}u(x,y)=\lim_{y\to 0-}u(x,y), \;\; 0\le x\le 1,$$

$$\lim_{y\to 0+} y^{1-\gamma}\big(y^{(1-\gamma)(1-\delta)}u(x,y)\big)_y=\lim_{y\to 0-}(-y)^{\alpha}u_y(x,y), \;\; 0<x<1,$$

where $\beta=\dfrac{2m-1+2\alpha}{2(2m+1)}$, A_1, A_2 are real constants, $g(x)$ is given function such that $g(x)\in C^1(\overline{I})\cap C^2(I)$,

$$\Theta(x)=\left(\frac{x_0}{2};-\Big(\frac{2m+1}{4}x_0\Big)^{\frac{2}{2m+1}}\right)$$

is the point of intersection of the characteristic OC with the characteristic issuing from the point $x_0\in I$.

We will seek a solution $u(x,y)$ of the given problem in the class of functions twice differentiable in the domain D such that

$$y^{(1-\gamma)(1-\delta)}u(x,y)\in C(\overline{D^+}), \; u(x,y)\in C(\overline{D^-}),$$

$$y^{(1-\gamma)}(y^{(1-\gamma)(1-\delta)}u(x,y))_y\in C(D^+\cup\{(x,y):0<x<1,\,y=0\}),$$

$$u_{xx}\in C(D^+\cup D^-), \quad u_{yy}\in C(D^-).$$

1 Main functional relations

Let's denote

$$\lim_{y\to 0+} y^{(1-\gamma)(1-\delta)}u(x,y)=\tau_1(x), \quad \lim_{y\to 0-}u(x,y)=\tau_2(x),$$

$$\lim_{y\to 0+} y^{1-\gamma}\big(y^{(1-\gamma)(1-\delta)}u(x,y)\big)_y=\nu_1(x), \quad \lim_{y\to 0-}(-y)^{\alpha}u_y(x,y)=\nu_2(x).$$

It is known [10] that a solution of equation (0.1) in domain D^+, satisfying condition (0.5) and the condition

$$\lim_{y\to 0+} y^{(1-\gamma)(1-\delta)}u(x,y)=\tau_1(x), \; \forall x\in\overline{I}$$

is given by the formula

$$u(x,y)=2\Gamma(\gamma+\delta(1-\gamma))\int\limits_0^1\tau_1(\xi)y^{(1-\delta)(1-\gamma)}\sum_{n=1}^{\infty}E_{\gamma,\gamma+\delta(1-\gamma)}(-k\lambda_n^2y^{\gamma})\sin(\lambda_n x)\sin(\lambda_n\xi)d\xi, \tag{1.1}$$

where $\lambda_n=n\pi$,

$$E_{\alpha,\beta}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)},$$

is two-parameter Mittag–Leffler function [11].

It is also known [4] that the functional relation between $\tau_1(x)$ and $\nu_1(x)$, derived from the parabolic part D^+ on the line $y=0$, has the form

$$\nu_1(x) = \frac{k\gamma\Gamma(\gamma+\delta(1-\gamma))}{\Gamma(2\gamma+\delta(1-\gamma))}\tau_1''(x),\ 0<x<1. \tag{1.2}$$

We now obtain the functional relation between $\tau_2(x)$ and $\nu_2(x)$ transferred to the line $y=0$ from the hyperbolic part D^- of domain D.

The solution of the modified Cauchy problem with initial data $u(x,0)=\tau_2(x),\ \ 0\le x\le 1$, $\lim\limits_{y\to -0}(-y)^{\frac{1-2m}{2}}u_y(x,y)=\nu_2(x),\ 0<x<1$, in domain D^- has the form [12]

$$u(x,y) = \frac{\Gamma(2\beta)}{\Gamma^2(\beta)}\int\limits_0^1 \tau_2\left(x+\frac{2(1-2t)}{2m+1}(-y)^{\frac{2m+1}{2}}\right)t^{\beta-1}(1-t)^{\beta-1}dt-$$

$$-\frac{2}{2m+1}\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}(-y)^{1-\alpha}\int\limits_0^1 \nu_2\left(x+\frac{2(1-2t)}{2m+1}(-y)^{\frac{2m+1}{2}}\right)t^{-\beta}(1-t)^{-\beta}dt. \tag{1.3}$$

Let us determine $u[\Theta(x)]$ by using formula (1.3)

$$u[\Theta(x)] = \frac{\Gamma(2\beta)}{\Gamma^2(\beta)}\int\limits_0^x \frac{x^{1-2\beta}\xi^{\beta-1}\tau_2(\xi)d\xi}{(x-\xi)^{1-\beta}}-$$

$$-\frac{2}{2m+1}\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}\Big(\frac{2m+1}{4}x\Big)^{\frac{2(1-\alpha)}{2m+1}}\int\limits_0^x \frac{x^{2\beta-1}\xi^{-\beta}\nu_2(\xi)d\xi}{(x-\xi)^{\beta}},$$

or, for brevity, using relation (0.2)

$$u[\Theta(x)] = k_1 I_{0+}^{\beta,0,\beta-1}\tau_2(x) + k_2 I_{0+}^{1-\beta,2\beta-1,\beta-1}\nu_2(x), \tag{1.4}$$

where $k_1=\dfrac{\Gamma(2\beta)}{\Gamma(\beta)},\ k_2=-\dfrac{\Gamma(1-2\beta)}{2\Gamma(1-\beta)}\Big(\dfrac{2m+1}{4}\Big)^{-2\beta}$.

Substituting the value found in (1.4) into condition (0.5), and relying on the composition formula (see, [9], p. 327)

$$\big(I_{0+}^{\alpha,\beta,\eta}I_{0+}^{\gamma,\delta,\alpha+\beta}f\big)(x) = \big(I_{0+}^{\alpha+\gamma,\beta+\delta,\eta}f\big)(x),\ (\gamma>0),$$

and considering (0.3) and (0.4), we obtain

$$(A_1k_1+A_2)\tau_2(x) + A_1k_2(D_{0+}^{2\beta-1}\nu_2)(x) = g(x). \tag{1.5}$$

(1.6) is relationship between $\tau_2(x)$ and $\nu_2(x)$, transferred from the hyperbolic part D^- of the domain D onto the line $y=0$.

Differentiating both sides of (1.6) twice with respect to x and taking into account

$$\big(D_{0+}^{\alpha}\varphi\big)(x) = \big(I_{0+}^{-\alpha,\alpha,\eta}\varphi\big)(x) = \Big(\frac{d}{dx}\Big)^n\big(I_{0+}^{n-\alpha}\varphi\big)(x),$$

we obtain

$$(A_1k_1 + A_2)\tau_2''(x) + A_1k_2(D_{0+}^{1+2\beta}\nu_2)(x) = g''(x). \tag{1.6}$$

2 Existence result

Assuming $\tau_1(x) = \tau_2(x) = \tau(x)$ and $\nu_1(x) = \nu_2(x) = \nu(x)$, and using (1.6) and (1.2), we obtain a fractional differential equation of order $1 + 2\beta$:

$$\left(D_{0+}^{1+2\beta}\nu\right)(x) - \lambda\nu(x) = h(x),$$

where

$$\lambda = -\frac{A_1k_1 + A_2}{A_1k_2}\frac{\Gamma(2\gamma + \delta(1-\gamma))}{k\gamma\Gamma(\gamma + \delta(1-\gamma))}, \quad h(x) = \frac{1}{A_1k_2}g''(x).$$

It is known (see, [9], pp. 601–602) that the general solution of a fractional differential equation of order $1 < 1 + 2\beta < 2$ is given by

$$\nu(x) = c_1x^{2\beta}E_{1+2\beta,1+2\beta}(\lambda x^{1+2\beta}) + c_2x^{2\beta-1}E_{1+2\beta,2\beta}(\lambda x^{1+2\beta}) + \\ + \int\limits_0^x (x-t)^{2\beta}E_{1+2\beta,1+2\beta}(\lambda(x-t)^{1+2\beta})h(t)dt, \tag{2.1}$$

where c_1 and c_2 are arbitrary constants.

We substitute (2.1) into (1.6) and deduce

$$\tau(x) = c_1^*\left(I_{0+}^{1-2\beta}t^{2\beta}E_{1+2\beta,1+2\beta}(\lambda t^{1+2\beta})\right)(x) + c_2^*\left(I_{0+}^{1-2\beta}t^{2\beta-1}E_{1+2\beta,2\beta}(\lambda t^{1+2\beta})\right)(x) + \\ + k_3\left(I_{0+}^{1-2\beta}\int\limits_0^t (t-s)^{2\beta}E_{1+2\beta,1+2\beta}(\lambda(t-s)^{1+2\beta})h(s)ds\right)(x) + \frac{1}{A_1k_1 + A_2}g(x), \tag{2.2}$$

where

$$k_3 = -\frac{A_1k_2}{A_1k_1 + A_2}, \; c_1^* = k_3c_1, \; c_2^* = k_3c_2.$$

To simplify expression (2.2), we use two lemmas proved in work [13].

Lemma 2.1. *If* $0 < \beta < \frac{1}{2}$, $\lambda \in \mathbb{C}$, *then*

$$(I_{0+}^{1-2\beta}t^{2\beta}E_{1+2\beta,1+2\beta}(\lambda t^{1+2\beta}))(x) = xE_{1+2\beta,2}(\lambda x^{1+2\beta})$$

and

$$(I_{0+}^{1-2\beta}t^{2\beta-1}E_{1+2\beta,2\beta}(\lambda t^{1+2\beta}))(x) = E_{1+2\beta,1}(\lambda x^{1+2\beta}) = E_{1+2\beta}(\lambda x^{1+2\beta}).$$

Lemma 2.2. *If* $0 < \beta < \frac{1}{2}$, $\lambda \in \mathbb{C}$, *then*

$$\left(I_{0+}^{1-2\beta}\int\limits_0^t (t-s)^{2\beta}E_{1+2\beta,1+2\beta}(\lambda(t-s)^{1+2\beta})h(s)ds\right)(x) = \\ = \int\limits_0^x (x-s)E_{1+2\beta,2}(\lambda(x-s)^{1+2\beta})h(s)ds.$$

Then, from equality (2.2) we obtain a formula for $\tau(x)$:

$$\tau(x) = c_1^* x E_{1+2\beta,2}(\lambda x^{1+2\beta}) + c_2^* E_{1+2\beta}(\lambda x^{1+2\beta}) +$$
$$+k_3 \int_0^x (x-s) E_{1+2\beta,2}(\lambda (x-s)^{1+2\beta}) h(s) ds + \frac{1}{A_1 k_1 + A_2} g(x). \tag{2.3}$$

To determine the constants c_1^* and c_2^*, we can use the relations $\tau(0) = \tau(1) = 0$. Substituting $x = 0$ into (2.3) and taking into account the equality $E_{1+2\beta}(0) = 1$ [14], we obtain

$$c_2^* = -\frac{1}{A_1 k_1 + A_2} g(0). \tag{2.4}$$

Substituting $x = 1$ into formula (2.3) and taking into account expression (2.4), we find

$$c_1^* = \frac{1}{E_{1+2\beta}(\lambda)} \Big(\frac{1}{A_1 k_1 + A_2} \big(E_{1+2\beta,2}(\lambda) g(0) - g(1) \big) - k_3 \int_0^1 (1-s) E_{1+2\beta,2}(\lambda (1-s)^{1+2\beta}) h(s) ds \Big).$$

Thus, substituting (2.3) into formula (1.1) (with $\tau_1 = \tau$), we obtain an explicit solution $u(x, y)$ of the problem under consideration.

Conclusion

In this work, a boundary value problem involving the generalized M. Saigo operator for a mixed-type equation with the Hilfer fractional derivative is studied. To establish the existence of a solution, the problem is reduced to an ordinary fractional differential equation. Moreover, the results obtained in this paper provide a basis for future investigations of nonlocal boundary value problems and may be applied to the analysis of more complex models with fractional derivatives and nonlocal conditions.

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