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Density-Type Properties of the Space of the Solutions of an Ordinary Differential Equation

Farkhod G. Mukhamadiev
National University of Uzbekistan, 4, Univeritet Street, Tashkent, Ozbekistan
Kimyo International University in Tashkent, 156, Usman Nasyr street, Tashkent, Ozbekistan
E-mail: farhod8717@mail.ru, https://orcid.org/0000-0002-1892-4899

Zhyldyz A. Artykova
Osh State University, 331, Lenin street, Osh, Kyrgyzstan
E-mail: jartykova@oshsu.kg, https://orcid.org/0009-0005-6513-9366

Tursun K. Yuldashev*
Tashkent State Transport University, 1, Temiryo'lchilar Street, Tashkent, Ozbekistan
V. I. Romanovskii Institute of Mathematics of Ozbekistan Academy of Science, Tashkent, Ozbekistan
Osh State University, 331, Lenin street, Osh, Kyrgyzstan
E-mail: tursun.k.yuldashev@gmail.com, https://orcid.org/0000-0002-9346-5362

Gulbadan M. Matieva
Osh State University, 331, Lenin street, Osh, Kyrgyzstan
E-mail: gulbadan_57@mail.ru

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Abstract. In this paper we provide a detailed analysis of the density-type properties of the space of solutions of an ordinary differential equation. We show that the density, local density, weak density, and local weak density of the space of solutions of an ordinary differential equation are countable. Furthermore, we prove that these properties are preserved under functors of hyperspace and superextensions of the space of solutions of an ordinary differential equation.

Key words: *Density, local density, weak density, local weak density, hyperspace, superextension, ordinary differential equation.*

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Introduction and Preliminaries

The theory of ordinary differential equations is the most interesting section of mathematical science for research and application (see, for example, [1]–[6]). The study of cardinal properties of the solution space of differential equations is relevant.

There are a lot of publications of studying on density-type properties of topological spaces, are appeared [7]–[11]. The interest in the study of cardinal invariants of topological spaces is high among mathematicians [12]–[14].

Cardinal function theory constitutes a strong area within the mathematical sciences, offering a wide range of practical applications. Many real-world problems can be reduced to finding a cardinal function, along with the topological invariants. A wide range of analytical and numerical methods has been developed to solve such problems effectively across disciplines in science and engineering (see, [15], [16]).

In this paper, problems of the theory of cardinal invariants of the space of solutions of an ordinary differential equation under functor's hyperspaces and superextensions are considered.

A set $A \subset X$ is dense in X if $\overline{A} = X$. The density is defined as the smallest cardinal number of the form $|A|$, where A is a dense subset of X. This cardinal number is denoted by $d(X)$. If $d(X) \leq \aleph_0$, then we say that the space X is separable [16].

We say that the local density of a topological space X is τ at a point x, if τ is the smallest cardinal number such that x has a neighborhood of density τ in X. The local density at a point x is denoted by $ld(x)$. The local density of a topological space X is defined as the supremum of all numbers $ld(x)$ for $x \in X$: $ld(X) = \sup\left\{ld(x):\ x \in X\right\}$ [7, 8]. It is known that, for any topological space we have $ld(X) \leq d(X)$.

We say that the weak density of the topological space is $\tau \geq \aleph_0$, if τ is the smallest cardinal number such that there exists a π-base coinciding with τ of centered systems of open sets, i.e. there is a π-base $B = \cup\left\{B_\alpha : \alpha \in A\right\}$, where B_α is a centered system of open sets $\alpha \in A$, such that $|A| = \tau$. Weak density of topological space X is denoted by $wd(X)$ [7].

Topological space X is said local weak τ-dense at a point x, if τ is the smallest cardinal number such that x has a neighborhood of weak density τ in X. Local weak density at a point x is denoted by $lwd(x)$. The local weak density of a topological space X is defined as the supremum of all numbers $lwd(x)$ for $x \in X$: $lwd(X) = \sup\left\{lwd(x) : x \in X\right\}$ [7], [8].

Let X be a T_1-space. The collection of all nonempty closed subsets of X we denote by $\exp X$. The family B of all sets in the form

$O\langle U_1,\ldots,U_n\rangle = \left\{F : F \in \exp X,\ F \subset \bigcup\limits_{i=1}^{n} U_i,\ F \cap U_i \neq \varnothing,\ i = 1,2,...,n\right\}$, where $U_1,\ldots,U_n$ is a sequence of open sets of X, generates the topology on the set $\exp X$. This topology is called the Vietoris topology. The $\exp X$ with the Vietoris topology is called the exponential space or the hyperspace of X [15]. Let X be a T_1-space. Denote by $\exp_{\mathrm{n}} X$ the set of all closed subsets of X cardinality of that is not greater than the cardinal number n, i.e. $\exp_{\mathrm{n}} X = \left\{F \in \exp X : |F| \leq n\right\}$.

Let's put $\exp_\omega X = \cup\left\{\exp_{\mathrm{n}} X : n = 1,2,...\right\}$, $\exp_c X = \left\{F \in \exp X : F \text{ is compact in } X\right\}$. It is clear, that $\exp_{\mathrm{n}} X \subset \exp_\omega X \subset \exp_c X \subset \exp X$ for any topological space X.

A system $\xi = \left\{F_\alpha :\ \alpha \in A\right\}$ of closed subsets of a space X is called linked, if any two elements of ξ intersect. Any linked system can be upgraded to a maximum linked system (MLS). But such upgrade, as a rule, is not one valued. A linked system of space is MLS, if and only if it possesses the following completeness property [15]: "If a closed set $A \subset X$ intersects with every element of ξ, then $A \in \xi$". We denote λX as the set of all MLS of the space X. For the closed set $A \subset X$ we consider $A^+ = \left\{\xi \in \lambda X :\ A \in \xi\right\}$. For the open set $U \subset X$ we consider $O(U) = \left\{\xi \in \lambda X : \text{there exists } F \in \xi \text{ such that } F \subset U\right\}$.

The family of sets of the form $O(U)$ covers the set λX ($O(X) = \lambda X$). So, it forms an open prebase of the topology on λX. The set λX, equipped with this topology, is called as the superextension of the space X [5]. Let X be topological space and λX be its superextension. MLS $\xi \in \lambda X$ is called compact, if it contains at least one compact element, and is denoted by CMLS. The space $\lambda_c X = \{\xi \in \lambda X : \ \xi \text{ is CMLS}\}$ we call as compact super kernel (or compact superextension) of the topological space X. It is clear that $\lambda_c X \subset \lambda X$. We see that $\lambda^* X \subseteq \lambda_c X \subseteq \lambda X$ for topological T_1-space X. If the space X is compact, then we have the equality $\lambda_c X = \lambda X$. If the space X is discrete, then we have another equality $\lambda^* X = \lambda_c X$. The basement of the CMLS ξ in X is the family $\mathfrak{F}(\xi) = \{F \in \xi : \ F \text{ is a compact}\}$.

Let us have a differential equation $y'(t) = f(t, y(t))$, where the function f is continuous in a domain U. We call a function $z \in C_s(U)$ a solution of this differential equation, if a) its domain consists of one point or b) the function z is defined on a certain interval $[a, b]$, where at each point t of this interval the derivative z' exists and is equal to $f(t, z(t))$. Of course, assigning a function to solutions in accordance with a) is a certain artificial device. However we can't do without it, since otherwise at each step of our construction we would have to make the appropriate reservations. Let Z be the set of all solutions of our differential equation defined in this way. What properties of this set are used when discussing the topological properties of solutions of the equation? Let us list the main ones.

1. If the function z belongs to the set Z and the segment I lies in the domain of definition of the function z, then $z|_I \in Z$.

2. If the intersection I of the domains of definition of functions $z,\ z_1, z_2 \in Z$ is nonempty and $z_1|_I = z_2|_I$, then a function defined on a segment $\pi(z_1) \cap \pi(z_2)$:

$$z(t) = \begin{cases} z_1(t), & \text{at } \ t \in \pi(z_1), \\ z_2(t), & \text{at } \ t \in \pi(z_2) \end{cases}$$

also belongs to the set Z.

3. The set Z contains all functions belonging to $C_s(U)$, the domain of the definitions of which are one-point, and if the function $z \in C_s(U)$ given on a certain segment $[a, b]$, where $a < b$, satisfies the condition: *for any segment I lying in the interval (a, b), $z|_I \in Z$, then* $z \in Z$. The solution set Z satisfies Conditions 1 and 2 and the first part of Condition 3 follows immediately from our definition of a solution. Satisfaction of the second part of Condition 3 can be proven. For example, we use Lagrange's formula, which is well-known from introductory mathematical analysis. We now turn to those properties of the solution set that form the content of the first fundamental theorems of the theory of ordinary differential equations.

4. For any point (t, y) in the domain U, there exists a function z, belonging to the set Z, defining on some interval. The interval contains the point t within itself, and taking the value y at the point t. This property easily aligns with the property involved in the solution existence theorem.

5. For any compact subset K of U, the set Z_K is compact. Here and below, for $M \subset U$ and $Z \subset C_s(U)$ we denote $Z_M = \{z : z \in Z,\ Gr(z) \subset M\}$. Compactness implies the existence of limit points for infinite sets. However, from the elements of an infinite set, one can select a sequence (of pairwise distinct elements), and, therefore, compactness is equivalent (in the case of a metric space) to the ability to select a convergent subsequence from an arbitrary sequence of points. This property of solution sequences (under appropriate additional assumptions) is constantly used in the theory of ordinary differential equations. Condition 5, which we have presented, is advantageous in that, when formulated concisely, it has a zone of immediate validity

significantly broader than ordinary conditions of this type.

6. If the domains of definition of the functions $z_1, z_2 \in Z$ coincide and at some point t of their common domain of definition $z_1(t) = z_2(t)$, then $z_1 = z_2$ (i.e., the functions coincide over the entire domain of definition). Here it is easy to recognize the property that appears in the uniqueness theorem.

7. For any point (t, y) of the domain U there is a number $\delta > 0$ such that for any point $s \in (t-\delta, t+\delta)$ the set $\{z(s) : z \in Z,\ s, t \in \pi(z),\ z(t) = y(t)\}$ is connected.

For some undefined or related concepts, we refer the reader to [2] and [15]–[22].

1 Main results

Let $I = [a, b] \subset \mathbb{R}$ be compact and consider the ordinary differential equation

$$x'(t) = f(t, x(t)), \quad t \in I, \tag{1.1}$$

where $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and locally Lipschitz with respect to x.

Fix $t_0 \in I$. For each $x_0 \in \mathbb{R}^n$, the initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0$$

admits a unique solution $x(\cdot, x_0) \in C(I, \mathbb{R}^n)$.

Define

$$\mathscr{S} = \{x(\cdot, x_0) : x_0 \in \mathbb{R}^n\} \subset C(I, \mathbb{R}^n),$$

endowed with the subspace topology induced by the supremum norm.

Lemma 1.1. *The map*

$$\Phi : \mathbb{R}^n \to C(I, \mathbb{R}^n), \quad \Phi(x_0) = x(\cdot, x_0),$$

is continuous.

Proof. Let $x_0, y_0 \in \mathbb{R}^n$ and denote $u(t) = x(t, x_0)$, $v(t) = x(t, y_0)$. Then

$$\|u(t) - v(t)\| \le \|x_0 - y_0\| + \int_{t_0}^{t} L\|u(s) - v(s)\|\, ds,$$

where $0 < L = \text{const}$ is a Lipschitz constant of f on a suitable compact set. By Grönwall's inequality, we obtain

$$\|u(t) - v(t)\| \le \|x_0 - y_0\| e^{L|t-t_0|}.$$

Taking supremum over $t \in I$ yields

$$\|\Phi(x_0) - \Phi(y_0)\|_\infty \le C\|x_0 - y_0\|.$$

Hence follows that Φ is continuous. □

Lemma 1.2. *The mapping Φ is injective.*

Proof. If $\Phi(x_0) = \Phi(y_0)$, then $x(t_0, x_0) = x(t_0, y_0)$. Hence, we get $x_0 = y_0$ by uniqueness of solutions. □

Corollary 1.1. *$\mathscr{S}$ is homeomorphic to $\Phi(\mathbb{R}^n)$.*

Lemma 1.3. *$\mathbb{R}^n$ has countable density.*

Proof. The set $\mathbb{Q}^n$ is countable and dense in $\mathbb{R}^n$. □

Theorem 1.1.

$$d(\mathscr{S}) = ld(\mathscr{S}) = \aleph_0.$$

Proof. By continuity of Φ, the image of a dense set is dense in the image. Thus $\Phi(\mathbb{Q}^n)$ is dense in $\mathscr{S}$, so $d(\mathscr{S}) \leq \aleph_0$. Since $\mathscr{S}$ is infinite, its density cannot be finite.

As $\mathscr{S}$ is metrizable, local density coincides with density at every point. Hence $ld(\mathscr{S}) = \aleph_0$. □

Theorem 1.2.

$$wd(\mathscr{S}) = lwd(\mathscr{S}) = \aleph_0.$$

Proof. In metrizable spaces, the weak topology generated by continuous real-valued functions coincides with the original topology. Therefore weak closure equals ordinary closure. Thus, weak density equals density, and the same holds locally. □

Theorem 1.3. *If $\mathscr{S}$ is a space of solutions of an ordinary differential equation, then*

$$d(\exp_c(\mathscr{S})) = d(\mathscr{S}).$$

Proof. Assume that $d(\mathscr{S}) = \kappa$. Let $D \subset \mathscr{S}$ be a dense subset with $|D| = \kappa$. Consider the family

$$\mathscr{D} = \{K \subset D : K \text{ is nonempty and finite}\}.$$

Since D is dense in $\mathscr{S}$, every nonempty open set in $\exp_c(\mathscr{S})$ contains some finite subset of D. Consequently, $\mathscr{D}$ is dense in $\exp_c(\mathscr{S})$. Moreover, we have $|\mathscr{D}| = \kappa$. Hence, we obtain $d(\exp_c(\mathscr{S})) \leq \kappa$.

Conversely, since $\mathscr{S}$ embeds homeomorphically into $\exp_c(\mathscr{S})$ via $x \mapsto \{x\}$, we have $d(\mathscr{S}) \leq d(\exp_c(\mathscr{S}))$. Therefore $d(\exp_c(\mathscr{S})) = d(\mathscr{S})$. □

Corollary 1.2. *For the solution space $\mathscr{S}$,*

$$d(\exp_c(\mathscr{S})) = \aleph_0.$$

Theorem 1.4. *If $\mathscr{S}$ is a space of solutions of an ordinary differential equation, then*

$$ld(\exp_c(\mathscr{S})) = ld(\mathscr{S}).$$

Proof. Let $K \in \exp_c(\mathscr{S})$. Every basic neighborhood of K in the Vietoris topology is determined by finitely many open neighborhoods in $\mathscr{S}$. Since $\mathscr{S}$ is metrizable, each such neighborhood has density at most $ld(\mathscr{S})$. Finite products and finite unions preserve local density bounds. Hence

$$ld(K, \exp_c(\mathscr{S})) \leq ld(\mathscr{S}).$$

Conversely, for any $x \in \mathscr{S}$, the embedding $x \mapsto \{x\}$ preserves neighborhoods. Thus $ld(\mathscr{S}) \leq ld(\exp_c(\mathscr{S}))$. □

Corollary 1.3.

$$ld(\exp_c(\mathscr{S})) = \aleph_0.$$

Theorem 1.5. *If X is metrizable, then*

$$wd(\exp_c(X)) = d(\exp_c(X)), \quad lwd(\exp_c(X)) = ld(\exp_c(X)).$$

Proof. In metrizable spaces, the weak topology generated by continuous real-valued functions coincides with the original topology. Since $\exp_c(X)$ is metrizable whenever X is, weak closure equals ordinary closure. Thus, weak density and local weak density coincide with their classical counterparts. □

Corollary 1.4.

$$wd(\exp_c(\mathscr{S})) = lwd(\exp_c(\mathscr{S})) = \aleph_0.$$

Theorem 1.6. *If X is separable metrizable, then*

$$d(\lambda X) = d(X).$$

Proof. The superextension λX is a continuous image of $\exp_c(\exp_c(X))$. By Theorem 1.3,

$$d(\exp_c(\exp_c(X))) = d(X).$$

Since density does not increase under continuous images, we obtain $d(\lambda X) \leq d(X)$. The natural embedding of X into λX implies the reverse inequality. □

Theorem 1.7. *If X is metrizable, then*

$$ld(\lambda X) = ld(X).$$

Proof. Let $\xi \in \lambda X$. Neighborhoods of ξ are determined by finitely many open sets in X. Using metrizability, each such neighborhood contains a dense subset of size at most $ld(X)$. Thus $ld(\lambda X) \leq ld(X)$. The canonical embedding of X into λX yields the opposite inequality. □

Theorem 1.8. *If X is metrizable, then*

$$wd(\lambda X) = lwd(\lambda X) = d(X).$$

Proof. Since λX is metrizable whenever X is, weak and ordinary topologies coincide. Thus weak density invariants reduce to classical density invariants, which are preserved by Theorems 1.6 and 1.7. □

Corollary 1.5. *For the solution space $\mathscr{S}$,*

$$d(\lambda\mathscr{S}) = ld(\lambda\mathscr{S}) = wd(\lambda\mathscr{S}) = lwd(\lambda\mathscr{S}) = \aleph_0.$$

Conclusion

We study density-type properties of the space of solutions of ordinary differential equations with continuous right-hand side. We proved that, the density, local density, weak density and local weak density of the space of solutions of ordinary differential equations are countable. Also, we proved that the exponential functor $\exp$ is preserve the density, local density, weak density and local weak density of the space of solutions of ordinary differential equations. Besides, proved that the functor superextension λ is also preserve the density, local density, weak density and local weak density of the space of solutions of ordinary differential equations. We will need this work in our future work to investigate cardinal invariants and functorial properties of the space of solutions of ordinary differential equations.

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