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Nonlinear Fractional-Differential-Integral Equation with Product of Two Nonlinear Functions, Degeneration and Maxima

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Abstract. *In this article a nonlinear initial and final values problem for a Gerasimov–Caputo type fractional differential equation with degeneration is considered in the case of differentiation order is* $0 < \alpha \leq 1$. *The right-hand side of the equation consists product of two nonlinear functions, Fredholm integral term and construction of maxima from unknown function. The solution of this fractional differential-integral equation is studied in the Banach space. A nonlinear integral equation is obtained by the aid of Mittag–Leffler function. The method of successive approximations in combination with the method of contracting mapping is applied in proof of one valued solvability of the problem. The continuous dependence of solution of the problem on initial data also is studied.*

Key words: *Initial and final values problem, Gerasimov–Caputo fractional equation, degeneration, product of two nonlinear functions, Mittag–Leffler function, method of contracting mapping.*

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Introduction. Statement of the Problem

Functional differential equations containing a product of two nonlinear functions arise when solving problems of nonlinear mechanics and nonlinear optimal control. Of theoretical interest is the study of nonlocal problems for functional-differential equations. By today a lot of publications of studying nonlocal problems for differential equations, describing many natural and practical processes, are appeared (see, [1]–[11]).

Some problems of mechanics turn out to be initial-boundary (mixed) problems. Many mixed problems are studied in solving different problems of hydrodynamics [12]. In [13], [14] mixed problems for linear differential equations of parabolic and hyperbolic types were studied. In works [15, 16], mixed problems for nonlinear differential and integro-differential equations of the second, fourth and higher orders were studied. The main equations of the theory of non-stationary filtration in fractured-pore formations are formulated in the work of G. I. Barenblatt, Yu. P. Zheltov and I. N. Kochina [17] (see also [18]) and, further, developed by many authors [19]–[24]. The theory and applications of fractional calculus have been developed by many authors (see, for example, [25]–[33]). Investigation of the well-known fractional order differentiation operators of Riemann–Liouville type and Gerasimov–Caputo type are important, when they describe diffusion processes [26, vol. 1, 47–85]. A physical and engineering applications of the generalized fractional operators are given in [26, vol. 4–8]. Note that some boundary value conditions take place in modeling problems of the flow around a profile by a subsonic velocity stream with a supersonic zone. Different mixed and boundary value problems for differential and integro-differential equations with identification source were studied in the works of many authors (see, for example, in the works [34]–[54]).

The present paper is further development of the work [55]. On the segment $[0,T]$ the fractional-differential-integral equation of the following form is considered

$$ {}_C D_{0t}^{\alpha} x(t) + t^{\beta} x(t) = \int\limits_0^T l\big(s, x(s), \max\{x(\tau) : \tau \in [s-h_1, s]\}\big) ds+ $$

$$ +g\big(t, x(t)\big) f\big(t, x(t), \max\{x(\tau) : \tau \in [t, t+h_2]\}\big) \tag{0.1} $$

with initial and final values conditions

$$ \left[\begin{array}{ll} x(0) = \varphi_0 = \text{const}, & \\ x(\eta) = \varphi_1(\eta), & \varphi_1(\eta) \in C[-h_1, 0], \\ x(\xi) = \varphi_2(\xi), & \varphi_2(\xi) \in C[T, T+h_2], \end{array} \right. \tag{0.2} $$

where β, T are given positive real numbers. It is well known that for the fractional order $0 < \alpha \le 1$ correspondents the operator

$$ {}_C D_{0t}^{\alpha} \eta(t) = J_{0t}^{1-\alpha} \eta'(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{\eta'(s)}{(t-s)^{\alpha}} ds, \quad {}_C D_{0t}^{1} \eta(t) = \eta'(t), \quad t \in (0,T), $$

which is called as Gerasimov–Caputo type fractional operator,

$$ J_{0t}^{\alpha} \eta(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{\eta(s) ds}{(t-s)^{1-\alpha}}, \quad t \in (0,T) $$

is Riemann–Liouville integral operator, $l(t, x(t), y(t)) \in C\big([0,T] \times X \times X, \mathbb{R}\big)$, $f\big(t, x(t), z(t)\big) \in C\big([0,T] \times X \times X, \mathbb{R}\big)$, $g\big(t, x(t)\big) \in C\big([0,T] \times X, \mathbb{R}\big)$, X is closed set in $\mathbb{R}$.

Problem 0.1. To find the function $x(t)$, which satisfies fractional-differential-integral equation (0.1), initial and final values conditions (0.2).

1 Construction of the Solution

The equation (0.1) rewrite as

$$ {}_C D_{0t}^{\alpha}\, x(t) = -t^{\beta} x(t) + p(t), \tag{1.1}$$

where we have denotation:

$$p(t) = \int_0^T l\big(s, x(s), \max\{x(\tau) : \tau \in [s-h_1, s]\}\big)ds+$$

$$+g\big(t, x(t)\big) f\big(t, x(t), \max\{x(\tau) : \tau \in [t, t+h_2]\}\big).$$

It is well known, that the two-parametric Mittag–Leffler function defined as

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha\, m + \beta)}, \quad z, \alpha, \beta \in \mathbb{C}, \quad \mathrm{Re}\,(\alpha) > 0.$$

The generalized Mittag–Lefler (Kilbas–Saigo) type function was defined for real $\alpha, m, l \in \mathbb{R}$ and complex $l \in \mathbb{C}$ by Kilbas and Saigo in the following form

$$E_{\alpha,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k, \;\; c_0 < 1, \;\; c_k = \prod_{j=1}^{k-1} \frac{\Gamma(\alpha[jm+l]+1)}{\Gamma(\alpha[jm+l+1]+1)}, \quad k = 1, 2, ...$$

These functions belong to the class of entire functions in the complex plane.

As an analog of the initial value problem (1.1), (0.2), we consider the following auxiliary equation

$$ {}_C D_{0t}^{\alpha} x(t) = -t^{\beta} x(t) + p(t), \quad x(0) = \varphi_0, \tag{1.2}$$

where $\beta, \varphi_0 \in \mathbb{R}$, $p(t) \in C([0,T], \mathbb{R})$.

Let $\gamma \in [0, 1)$. Then we consider the class of following functions ([27], [57]):

$$C_{\gamma}([0;T], \mathbb{R}) = \Big\{p(t) : t^{\gamma} p(t) \in C([0;T], \mathbb{R})\Big\},$$

$$C_{\gamma}^{\alpha}\big([0;T], \mathbb{R}\big) = \Big\{p(t) \in C\big([0;T], \mathbb{R}\big) : {}_C D_{0t}^{\alpha}\, p(t) \in C_{\gamma}\big([0;T], \mathbb{R}\big)\Big\}.$$

Lemma 1.1 ([25])**.** *Let $\gamma \in [0; \alpha]$, $\beta \geq 0$. Then for all $p(t) \in C_{\gamma}\big([0;T], \mathbb{R}\big)$ there exists a unique solution $x(t) \in C_{\gamma}^{\alpha}\big([0;T], \mathbb{R}\big)$ of the Cauchy problem (1.2). This solution has the following form*

$$x(t) = \varphi_0 E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left(t^{\alpha+\beta}\right) + \int_0^t K(t,s) p(s) ds, \tag{1.3}$$

where

$$K(t,s) = \sum_{i=1}^{\infty} K_i(t,s), \quad K_0(t,s) = \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1}, \tag{1.4}$$

$$K_i(t,s) = \frac{1}{\Gamma(\alpha)} \int_s^t \theta^{\beta} (t-\theta)^{\alpha-1} K_{i-1}(\theta, s) d\theta, \quad i = 1, 2, \; ... \tag{1.5}$$

$E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(t^{\alpha+\beta}\right)$ *is Kilbas–Saigo function.*

Lemma 1.2 ([56]). *For every $\alpha \in [0,1]$, $m > 0$ and $t \geq 0$ the estimate is true*

$$\frac{1}{1+\Gamma(1-\alpha)t} \leq E_{\alpha,m,m-1}(-t) \leq \frac{1}{1+\dfrac{\Gamma(1+\alpha(m-1))}{\Gamma(1+\alpha m)}t}.$$

Lemma 1.3 ([57]). *Let $\alpha < 2$, $z \in \mathbb{C}$, δ be real constant and σ be fixed number from the interval $\left(\frac{\pi\alpha}{2}, \min\{\pi, \pi\alpha\}\right)$. Then for $|\arg z| \leq \sigma$ and $|z| \geq 0$ the estimate is true*

$$|E_{\alpha,\mu}(z)| \leq M_1(1+|z|)^{\frac{1-\delta}{\alpha}} e^{\operatorname{Re} z^{\frac{1}{\alpha}}} + \frac{M_2}{1+|z|},$$

where M_1 and M_2 are constants, not depending from z.

Lemma 1.4 ([58]). *For the kernels* (1.4), $\gamma \in [0,\alpha]$, $\beta \geq 0$ *from* (1.5) *it is true that there holds the following estimate*

$$\left|K(t,s)\right| \leq M_3\,(t-s)^{\alpha-1}, \tag{1.6}$$

where $0 < M_3 = \text{const}$.

2 Main Results

The right-hand side of the integral equation (1.3) we write as a nonlinear integral operator

$$x(t) = J(t;x(t)) \equiv \varphi_0 E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-t^{\alpha+\beta}\right) +$$

$$+\int_0^t K(t,s)\int_0^T l\big(\theta, x(\theta), \max\{x(\tau) : \tau \in [\theta-h_1,\theta]\}\big)d\theta ds+$$

$$+\int_0^t K(t,s)\,g\big(s,x(s)\big)f\big(s,x(s),\max\{x(\tau) : \tau \in [s,s+h_2]\}\big)ds, \tag{2.1}$$

where

$$J(t;x(t)) : C([0,T];\mathbb{R}) \to C([0,T];\mathbb{R}),$$

By $C\big([0,T],\mathbb{R}\big)$ is denoted the Banach space with continuous function $x(t)$ on the segment $[0,T]$ and this space is equipped with the norm

$$\|x(t)\|_{C[0,T]} = \max_{0\leq t\leq T}|x(t)|.$$

We use also the vector space $BD\big([0,T],\mathbb{R}\big)$, which is Banach space with the following norm

$$\|x(t)\|_{BD[0,T]} = \|x(t)\|_{C[0,T]} + h\cdot\left\|x'(t)\right\|_{C[0,T]},$$

where $0 < h = \text{const}$.

Theorem 2.1. *Let the following conditions are fulfilled:*
1). $M_f = \max\limits_{0\leq t\leq T}|f(t,x(t),z(t))| < \infty$, $0 < M_f = \text{const}$;

2). $M_l = \max\limits_{0\le t\le T} |\, l(t, x(t), y(t)) \,| < \infty, \quad 0 < M_l = \text{const}$;

3). $M_g = \max\limits_{0\le t\le T} |\, g(t, x(t)) \,| < \infty, \quad 0 < M_g = \text{const}$;

4). *For all* $t \in [0, T]$, $x \in X$ *there exist such functions* $0 < a_f(t) \in C\big([0, T], \mathrm{R}\big)$, $\;0 < b_f(t) \in C\big([0, T], \mathbb{R}\big)$, $\;0 < c_f(t) \in C\big([0, T], \mathrm{R}\big)$ *that*

$$\big|\, f\big(t, x(t), z(t)\big) \,\big| \le a_f(t)\,|\,x(t)\,| + b_f(t)\,|\,z(t)\,| + c_f(t);$$

5). *For all* $t \in [0, T]$, $x \in X$ *there exist such functions* $0 < a_l(t) \in C\big([0, T], \mathbb{R}\big)$, $\;0 < b_l(t) \in C\big([0, T], \mathbb{R}\big)$, $\;0 < c_l(t) \in C\big([0, T], \mathbb{R}\big)$, *that*

$$\big|\, l\big(t, x(t), y(t)\big) \,\big| \le a_l(t)\,|\,x(t)\,| + b_l(t)\,|\,y(t)\,| + c_l(t);$$

6). *For all* $t \in [0, T]$, $x \in X$ *there exist such functions* $0 < a(t) \in C\big([0, T], \mathbb{R}\big)$, $0 < c_g(t) \in C\big([0, T], \mathbb{R}\big)$, *that*

$$\big|\, g\big(t, x(t)\big) \,\big| \le a_g(t)\,|\,x(t)\,| + c_g(t).$$

7). $\rho = \chi_1 + \chi_2 + \chi_3 < 1$, *where we denoted*

$$\chi_1 = M_3\,\frac{T^\alpha}{2\alpha}\left[\|\, a_l(t) \,\|_{C[0,T]} + M_f \|\, a_g(t) \,\|_{C[0,T]} + M_g \|\, a_f(t) \,\|_{C[0,T]}\right] < 1,$$

$$\chi_2 = M_3\,\frac{T^{\alpha+1}}{2\alpha}\;\|\, b_l(t) \,\|_{C[0,T]}\,, \quad \chi_3 = M_3\,\frac{T^\alpha}{2\alpha}\,\|\, b_f(t) \,\|_{C[0,T]}\,.$$

Then, if the solution of initial and final values problem (0.1), (0.2) *exists, there holds the following estimate*

$$\|\, x(t) \,\|_{C[0,T]} \le \frac{\chi_4}{1 - (\chi_1 + \chi_2 + \chi_3)}, \quad \chi_1 + \chi_2 + \chi_3 < 1,$$

where

$$\chi_4 = M_3\,|\,\varphi_0\,| + \frac{T^\alpha}{2\alpha} M_3 \left[T \,\|\, c_l(t) \,\|_{C[0,T]} + M_f \,\|\, c_g(t) \,\|_{C[0,T]} + M_g \,\|\, c_f(t) \,\|_{C[0,T]}\right],$$

Proof. Taking into account the conditions of the theorem and estimate (1.6), for the solution of the equation (2.1) we have estimate

$$\|\, x(t) \,\|_{C[0,T]} \le |\,\varphi_0\,| \max_{0\le t\le T} \left| E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}\left(-t^{\alpha+\beta}\right) \right| + \max_{0\le t\le T} \int\limits_0^t |\,K(t,s)\,| \int\limits_0^T \big|\, l\big(\theta, x(\theta), y(\theta)\big) \,\big|\, d\theta ds +$$

$$+\frac{1}{2}\max_{0\le t\le T} \int\limits_0^t |\,K(t,s)\,| \Big[\,\big|\, f\big(s, x(s), z(s)\big) \,\big|\; \big|\, g\big(s, x(s)\big) \,\big| + \big|\, f\big(s, x(s), z(s)\big) \,\big|\; \big|\, g\big(s, x(s)\big) \,\big|\,\Big] ds \le$$

$$\le M_3\,|\,\varphi_0\,| + M_3 \max_{0\le t\le T} \int\limits_0^t (t-s)^{\alpha-1} ds \int\limits_0^T \Big(\|\, a_l(t) \,\|_{C[0,T]}\; \|\, x(t) \,\|_{C[0,T]} +$$

$$+ \|\, b_l(t) \,\|_{C[0,T]}\; \|\, y(t) \,\|_{C[0,T]} + \|\, c_l(t) \,\|_{C[0,T]}\Big)\, dt +$$

$$+\frac{1}{2}M_3\max_{0\le t\le T}\int\limits_0^t(t-s)^{\alpha-1}ds\Big[\| f\big(t,x(t),z(t)\big)\|_{C[0,T]}\Big(\| a_g(t)\|_{C[0,T]}\| x(t)\|_{C[0,T]}+\| c_g(t)\|_{C[0,T]}\Big)+$$

$$+\| g\big(t,x(t)\big)\|_{C[0,T]}\Big(\| a_f(t)\|_{C[0,T]}\| x(t)\|_{C[0,T]}+\| b_f(t)\|_{C[0,T]}\| z(t)\|_{C[0,T]}+\| c_f(t)\|_{C[0,T]}\Big)\Big]\le$$

$$\le M_3|\varphi_0|+M_4M_3\Big[T\Big(\| a_l(t)\|_{C[0,T]}\| x(t)\|_{C[0,T]}+\| b_l(t)\|_{C[0,T]}\| y(t)\|_{C[0,T]}+\| c_l(t)\|_{C[0,T]}\Big)+$$

$$+\frac{1}{2}M_f\Big(\| a_g(t)\|_{C[0,T]}\| x(t)\|_{C[0,T]}+\| c_g(t)\|_{C[0,T]}\Big)+$$

$$+\frac{1}{2}M_g\Big(\| a_f(t)\|_{C[0,T]}\| x(t)\|_{C[0,T]}+\| b_f(t)\|_{C[0,T]}\| z(t)\|_{C[0,T]}+\| c_f(t)\|_{C[0,T]}\Big)\Big]\le$$

$$\le\chi_1\| x(t)\|_{C[0,T]}+\chi_2\| y(t)\|_{C[0,T]}+\chi_3\| z(t)\|_{C[0,T]}+\chi_4,$$

where

$$\chi_1=M_3\frac{T^\alpha}{\alpha}\Big[T\| a_l(t)\|_{C[0,T]}+M_f\| a_g(t)\|_{C[0,T]}+M_g\| a_f(t)\|_{C[0,T]}\Big],$$

$$\chi_2=M_3\frac{T^{\alpha+1}}{\alpha}\| b_l(t)\|_{C[0,T]},\quad \chi_3=M_3\frac{T^\alpha}{\alpha}\| b_f(t)\|_{C[0,T]},$$

$$\chi_4=M_3|\varphi_0|+\frac{T^\alpha}{\alpha}M_3\Big[T\| c_l(t)\|_{C[0,T]}+M_f\| c_g(t)\|_{C[0,T]}+M_g\| c_f(t)\|_{C[0,T]}\Big],$$

$$M_4=\max_{0\le t\le T}\int\limits_0^t(t-s)^{\alpha-1}ds\le\frac{T^\alpha}{\alpha},\quad y(t)=\max\{x(\tau):\tau\in[t-h_1,t]\},$$

$$z(t)=\max\big\{x(\tau):\tau\in\big[t,t+h_2\big]\big\}.$$

Hence, we obtain

$$\| x(t)\|_{C[0,T]}\le\chi_1\| x(t)\|_{C[0,T]}+\chi_2\| y(t)\|_{C[0,T]}+\chi_3\| z(t)\|_{C[0,T]}+\chi_4.$$

We denote $u(t)=\max\Big\{\| x(t)\|_{C[0,T]};\| y(t)\|_{C[0,T]};\| z(t)\|_{C[0,T]}\Big\}$, then from last inequality obtain

$$u(t)\le\big(\chi_1+\chi_2+\chi_3\big)u(t)+\chi_4. \tag{2.2}$$

As $\| x(t)\|_{C[0,T]}\le u(t)$, from (2.2) we get the proof of the theorem 2.1:

$$\| x(t)\|_{C[0,T]}\le\frac{\chi_4}{1-\big(\chi_1+\chi_2+\chi_3\big)},\quad \chi_1+\chi_2+\chi_3<1.$$

The theorem 2.1 is proved. □

Corollary 2.1. *If the conditions of the theorem 2.1 are fulfilled, then the problem* (0.1), (0.2) *has at least one solution on the segment* $[0,T]$.

Lemma 2.1 ([59])**.** *For the difference of two functions with maxima there holds the following estimate*

$$\|\max\{x(\tau):\tau\in[t-h_1,t]\}-\max\{y(\tau):\tau\in[t-h_1,t]\}\|_{C[0,T]}\le$$

$$\le\| x(t)-y(t)\|_{C[0,T]}+h_1\| x'(t)-y'(t)\|_{C[0,T]}=\| x(t)-y(t)\|_{BD[0,T]},$$

where $0 < h_1 = \text{const}$.

Analogously, it is true that

$$\left\| \max\left\{x(\tau) : \tau \in [t, t+h_2]\right\} - \max\left\{y(\tau) : \tau \in [t, t+h_2]\right\} \right\|_{C[0,T]} \leq$$

$$\leq \left\| x(t) - y(t) \right\|_{C[0,T]} + h_2 \left\| x'(t) - y'(t) \right\|_{C[0,T]} = \left\| x(t) - y(t) \right\|_{BD[0,T]}.$$

Theorem 2.2. *Let the conditions of the theorem 2.1 are fulfilled.*
1). *For all* $t \in [0, T]$, $x \in X$ *there exist such function* $0 < L_l = \text{const}$, *that*

$$\left| l\big(t, x_1, y_1\big) - l\big(t, x_2, y_2\big) \right| \leq L_l \left(\left| x_1 - x_2 \right| + \left| y_1 - y_2 \right| \right);$$

2). *For all* $t \in [0, T]$, $x \in X$ *there exist such function* $0 < L_g = \text{const}$, *that*

$$\left| g\big(t, x_1\big) - g\big(t, x_2\big) \right| \leq L_g \left| x_1 - x_2 \right|;$$

3). *For all* $t \in [0, T]$, $x \in X$ *there exist such function* $0 < L_f = \text{const}$, *that*

$$\left| f\big(t, x_1, z_1\big) - f\big(t, x_2, z_2\big) \right| \leq L_f \big(\left| x_1 - x_2 \right| + \left| z_1 - z_2 \right| \big);$$

4). $\rho = \max\left\{\gamma_{11};\, h\gamma_{21}\right\} < 1$, *where*

$$\gamma_{11} = M_3 \frac{T^\alpha}{\alpha} \big(2TL_l + M_f L_g + 2M_g L_f\big), \quad \gamma_{21} = M_{30} M_{40} \big(2TL_l + M_f L_g + 2M_g L_f\big).$$

Then the initial and final values problem (0.1), (0.2) *has a unique solution* $x(t) \in BD\big([0, T], \mathbb{R}\big)$, *which can be found from the iteration*

$$\left[\begin{array}{l} x^0(t) = \varphi_0 E_{\alpha,\, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left(-t^{\alpha+\beta}\right), \\ x^{m+1}(t) = J(t; x^m(t)), \;\; m = 0, 1, 2, \; \ldots \end{array} \right. \tag{2.3}$$

Proof. We use fixed point method and prove that the operator $J(t; x(t))$ on the right-side of the equation (2.1) is compressing in $C[0, T]$. For the zero approximation from the (2.3) we have

$$\left\| x^0(t) \right\|_{C[0,T]} \leq \left| \varphi_0 \right| \max_{0 \leq t \leq T} \left| E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left(-t^{\alpha+\beta}\right) \right| \leq M_3 \left| \varphi_0 \right|. \tag{2.4}$$

For arbitrary $m \in \mathbb{N}$ we obtain that

$$\left\| x^{m+1}(t) - x^m(t) \right\|_{C[0,T]} \leq$$

$$\leq \max_{0 \leq t \leq T} \int_0^t \left| K(t, s) \right| \int_0^T \left| l\big(\theta, x^m(\theta), y^m(\theta)\big) - l\big(\theta, x^{m-1}(\theta), y^{m-1}(\theta)\big) \right| d\theta ds +$$

$$+ \max_{0 \leq t \leq T} \int_0^t \left| K(t, s) \right| \left| f\big(s, x^m(s), z^m(s)\big) g\big(s, x^m(s)\big) - f\big(s, x^{m-1}(s), z^{m-1}(s)\big) g\big(s, x^{m-1}(s)\big) \right| ds \leq$$

$$\leq M_3 \frac{T^{\alpha+1}}{\alpha} \left\| l\big(t, x^m(t), y^m(t)\big) - l\big(t, x^{m-1}(t), y^{m-1}(t)\big) \right\|_{C[0,T]} +$$

$$+M_3\frac{T^\alpha}{\alpha}\left\| f\big(t,x^m(t),z^m(t)\big)\, g\big(t,x^m(t)\big)-f\big(t,x^m(t),z^m(t)\big)\, g\big(t,x^{m-1}(t)\big)\right\|_{C[0,T]}+$$

$$+M_3\frac{T^\alpha}{\alpha}\left\| f\big(t,x^m(t),z^m(t)\big)\, g\big(t,x^{m-1}(t)\big)-f\big(t,x^{m-1}(t),z^{m-1}(t)\big)\, g\big(t,x^{m-1}(t)\big)\right\|_{C[0,T]}\le$$

$$\le M_3\frac{T^{\alpha+1}}{\alpha}L_l\left(\left\| x^m(t)-x^{m-1}(t)\right\|_{C[0,T]}+\left\| y^m(t)-y^{m-1}(t)\right\|_{C[0,T]}\right)+$$

$$+M_3\frac{T^\alpha}{\alpha}M_fL_g\left\| x^m(t)-x^{m-1}(t)\right\|_{C[0,T]}+$$

$$+M_3\frac{T^\alpha}{\alpha}M_gL_f\left(\left\| x^m(t)-x^{m-1}(t)\right\|_{C[0,T]}+\left\| z^m(t)-z^{m-1}(t)\right\|_{C[0,T]}\right). \tag{2.5}$$

According to the lemma 2.1, we have

$$\left\| y^m(t)-y^{m-1}(t)\right\|_{C[0,T]}\le\left\| x^m(t)-x^{m-1}(t)\right\|_{C[0,T]}+h\left\| \dot x^m(t)-\dot x^{m-1}(t)\right\|_{C[0,T]},$$

$$\left\| z^m(t)-z^{m-1}(t)\right\|_{C[0,T]}\le\left\| x^m(t)-x^{m-1}(t)\right\|_{C[0,T]}+h\left\| \dot x^m(t)-\dot x^{m-1}(t)\right\|_{C[0,T]},$$

where $h=h_1+h_2$.

Taking into account last two estimates the inequality (2.5) we rewrite as

$$\left\| x^{m+1}(t)-x^m(t)\right\|_{C[0,T]}\le$$

$$\le M_3\frac{T^{\alpha+1}}{\alpha}L_l\left(2\left\| x^m(t)-x^{m-1}(t)\right\|_{C[0,T]}+h\left\| \dot x^m(t)-\dot x^{m-1}(t)\right\|_{C[0,T]}\right)+$$

$$+M_3\frac{T^\alpha}{\alpha}M_fL_g\left\| x^m(t)-x^{m-1}(t)\right\|_{C[0,T]}+$$

$$+M_3\frac{T^\alpha}{\alpha}M_gL_f\left(2\left\| x^m(t)-x^{m-1}(t)\right\|_{C[0,T]}+h\left\| \dot x^m(t)-\dot x^{m-1}(t)\right\|_{C[0,T]}\right)\le$$

$$\le\gamma_{11}\left\| x^m(t)-x^{m-1}(t)\right\|_{C[0,T]}+\gamma_{12}h\left\| \dot x^m(t)-\dot x^{m-1}(t)\right\|_{C[0,T]}, \tag{2.6}$$

where

$$\gamma_{11}=M_3\frac{T^\alpha}{\alpha}\left[2TL_l+M_fL_g+2M_gL_f\right],\quad \gamma_{12}=M_3\frac{T^\alpha}{\alpha}\big(TL_l+M_gL_f\big).$$

As $\gamma_{11}\ge\gamma_{12}$, from (2.6) we get

$$\left\| x^{m+1}(t)-x^m(t)\right\|_{C[0,T]}\le$$

$$\le\gamma_{11}\left(\left\| x^m(t)-x^{m-1}(t)\right\|_{C[0,T]}+h\left\| \dot x^m(t)-\dot x^{m-1}(t)\right\|_{C[0,T]}\right). \tag{2.7}$$

Taking into account that $K(t,t)=0$, we differentiate the equation (2.1):

$$\dot x(t)=J'(t;\dot x(t))\equiv\varphi_0\cdot\left[E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-t^{\alpha+\beta}\right)\right]'+$$

$$+\int\limits_0^t K'(t,s)\int\limits_0^T l\big(\theta,x(\theta),\max\{x(\tau):\tau\in[\theta-h_1,\theta]\}\big)d\theta ds+$$

$$+\int\limits_0^t K'(t,s)\, g\big(s,x(s)\big)f\big(s,x(s),\max\{x(\tau):\tau\in[s,s+h_2]\}\big)ds. \tag{2.8}$$

Analogously to the estimate (2.7), from (2.8) we have

$$\left\| \dot{x}^{m+1}(t) - \dot{x}^{m}(t) \right\|_{C[0,T]} \leq$$

$$\leq \gamma_{21} \left\| x^{m}(t) - x^{m-1}(t) \right\|_{C[0,T]} + \gamma_{22} h \left\| \dot{x}^{m}(t) - \dot{x}^{m-1}(t) \right\|_{C[0,T]}, \tag{2.9}$$

where

$$\gamma_{21} = M_{30} M_{40} \big(2TL_l + M_f L_g + 2M_g L_f \big), \quad \gamma_{22} = M_{30} M_{40} \big(TL_l + M_g L_f \big),$$

$$M_{30} = \max_{0 \leq t \leq T} \left| \left[E_{\alpha,\, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left(-t^{\alpha+\beta} \right) \right]' \right|, \quad M_{40} = \max_{0 \leq t \leq T} \int_{0}^{t} \left| K'(t,s) \right| ds.$$

As $\gamma_{21} \geq \gamma_{22}$, from (2.9) we have

$$\left\| \dot{x}^{m+1}(t) - \dot{x}^{m}(t) \right\|_{C[0,T]} \leq$$

$$\leq \gamma_{21} \Big(\left\| x^{m}(t) - x^{m-1}(t) \right\|_{C[0,T]} + h \left\| \dot{x}^{m}(t) - \dot{x}^{m-1}(t) \right\|_{C[0,T]} \Big). \tag{2.10}$$

We multiple the (2.10) to $0 < h$, and adding (2.7) and (2.10), we obtain

$$\left\| x^{m+1}(t) - x^{m}(t) \right\|_{BD[0,T]} \leq \rho \left\| x^{m}(t) - x^{m-1}(t) \right\|_{BD[0,T]}, \tag{2.11}$$

where

$$\rho = \max \big\{ \gamma_{11};\ h\gamma_{21} \big\} < 1.$$

From the estimates (2.4) and (2.11) implies that the operator $J(t; x(t))$ on the right-side of the equation (2.1) is contracting in the space $BD([0,T], \mathbb{R})$. Hence, we deduce that the nonlinear integral operator has a unique fixed point in the segment $[0,T]$. So, we proved that the initial and final values problem (0.1), (0.2) has a unique solution in the space $BD([0,T], \mathbb{R})$. □

Theorem 2.3. *Let be fulfilled the conditions of the Theorem 2.2. Then the solution of the problem* (0.1)*,* (0.2) *is continuously dependent from initial data in* (0.2)*.*

Proof. Let φ_{01} and φ_{02} are two values of the initial data φ_0 such that $|\varphi_{01} - \varphi_{02}| < \delta(\varepsilon)$, where $0 < \delta(\varepsilon)$ is small number and depends from the given small number $0 < \varepsilon$. Let $x_1(t)$ and $x_2(t)$ are two solutions of the nonlinear integral equation (2.1), corresponding to φ_{01} and φ_{02}. We will prove that $\left| x_1(t) - x_2(t) \right| < \varepsilon$.

So, from the integral equation (2.1) we have

$$\left\| x_1(t) - x_2(t) \right\|_{BD[0,T]} \leq M_3 \left| \varphi_{01} - \varphi_{02} \right| + \rho \cdot \left\| x_1(t) - x_2(t) \right\|_{BD[0,T]} <$$

$$< M_3 \delta(\varepsilon) + \rho \cdot \left\| x_1(t) - x_2(t) \right\|_{BD[0,T]}.$$

From last estimate we obtain the proof of the theorem

$$\left\| x_1(t) - x_2(t) \right\|_{BD[0,T]} < \frac{M_3}{1-\rho} \delta,$$

if we put that

$$\varepsilon = \frac{M_3}{1-\rho} \delta.$$

□

Conclusion

In this article a nonlinear initial and final values problem (0.1), (0.2) for a Gerasimov–Caputo type fractional-differential-integral equation with degeneration and maxima is considered in the case of order $0 < \alpha \leq 1$. The right-hand side of the equation (0.1) consists product of two nonlinear functions $f\big(t, x(t), y(t)\big) \cdot g\big(t, x(t)\big)$. The solution of this fractional differential equation is studied in the space $BD\big([0,T], \mathbb{R}\big)$. A nonlinear integral equation (2.1) is obtained by using the Mittag–Leffler function. In proof of one valued solvability of the problem the method of successive approximations in combination with the method of contracting mapping is applied. The continuous dependence of solution of the problem on initial data also is studied.

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