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Boundary Value Problem for a Seventh Order Nonhomogeneous Partial Differential Equation

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Abstract. *In this paper, we consider a boundary value problem for a seventh order partial differential equation. It is used Samarskii–Ionkin type boundary value conditions on spatial variable x. The non-self-adjoint spectral problem and adjoint spectral problem are studied. The systems of eigenvalues and eigenfunctions are determined. By the biorthogonal systems of eigenvalues, the Fourier series method of separation of variables is applied. Consequently, the unique solution of the boundary value problem is obtained in the form of Fourier series. Absolutely and uniformly convergence of Fourier series is proved.*

Key words: *Boundary value problem, seventh order equation, adjoint spectral problem, eigenfunctions, biorthogonal systems, unique solvability.*

MSC 2020: 35A01, 35A02, 35S15.

Introduction. Problem statement

Differential equations of mathematical physics have a wide range of applications in the sciences and technology (see, for examples [1]–[8]). According to applications this field of mathematics is developing at a rapid pace and a lot of works are being published (see, for examples [9]–[15]).

The main equations of the theory of non-stationary filtration in fractured-pore formations are formulated in the work of G. I. Barenblatt, Yu. P. Zheltov and I. N. Kochina [16] (see, also [17]) and, further, developed by many authors [18]–[23]. Boussinesq type partial differential equations with boundary conditions are studied partially, in [24]–[27].

In the works [28]–[30], the similar equation is studied by Cauchy and Dirichlet conditions. So, our new work, which presents in this paper, is further development of the works [28]–[30].

In this paper, we study the classical solvability of the boundary value problem for a seventh order partial differential equation. In the rectangular domain, $\Omega = \{0 < t < T,\, 0 < x < 1\}$ we consider the following partial differential equation

$$\left[\frac{\partial^3}{\partial t^3} + \frac{\partial^7}{\partial t^3 \partial x^4} + \omega^3 \frac{\partial^4}{\partial x^4}\right] U(t,x) = f(t,x), \tag{0.1}$$

where T is given positive number, ω is nonzero real parameter.

In solving partial differential equation (0.1) we use Samarskii–Ionkin type boundary value conditions

$$U(t,1) = 0, \quad U_{xx}(t,0) = 0, \quad U_x(t,0) = U_x(t,1), \quad U_{xxx}(t,0) = U_{xxx}(t,1), \quad 0 \le t \le T \tag{0.2}$$

and two-point boundary value conditions

$$U(0,x) = \varphi_1(x), \quad U(T,x) = \varphi_2(x), \quad U'(0,x) = \varphi_3(x), \tag{0.3}$$

where $\varphi_i(x)$ is enough smooth on the segment $[0,1]$.

Problem 0.1. To find a function $U(t,x) \in C(\overline{\Omega}) \cap C^{3,4}_{t,x}(\Omega)$, which satisfies differential equation (0.1) and the conditions (0.2), (0.3), where $\overline{\Omega} = \big\{0 \le t \le T,\, 0 \le x \le 1\big\}$.

1 Spectral Problems

We consider homogeneous differential equation

$$\frac{\partial U^3(t,x)}{\partial t^3} + \frac{\partial^7 U(t,x)}{\partial t^3\, \partial x^4} + \omega^3 \frac{\partial^4 U(t,x)}{\partial x^4} = 0. \tag{1.1}$$

We will look for a non-trivial particular solution of the equation in the form $U(t,x) = u(t) \cdot \vartheta(x)$. Substituting this product of functions, depending from different variables, into equation (1.1), we obtain

$$-\frac{u'''(t)}{u'''(t) + \omega^3 u(t)} = \frac{\vartheta^{(IV)}(x)}{\vartheta(x)}.$$

Hence, equating second fraction into λ we obtain

$$\vartheta^{(IV)}(x) - \lambda\vartheta(x) = 0, \quad \lambda \ge 0. \tag{1.2}$$

Using conditions (0.2), from product of two functions we obtain conditions for the eigenvalues λ and eigenfunctions $\vartheta(x)$:

$$\vartheta(1) = 0, \quad \vartheta''(0) = 0, \quad \vartheta'(0) = \vartheta'(1), \quad \vartheta'''(0) = \vartheta'''(1). \tag{1.3}$$

Solving the spectral problem (1.2), (1.3), we derive the eigenvalues [31]–[33]

$$\lambda_n = (2\pi n)^4, \quad n = 0, 1, 2, \ldots \tag{1.4}$$

Eigenfunctions, corresponding to the eigenvalues (1.4), have the forms

$$\vartheta_0(x) = 2(1-x), \quad \vartheta_{1n}(x) = -2\sin 2\pi n x, \quad \vartheta_{2n}(x) = \frac{e^{2\pi n x} - e^{2\pi n(1-x)}}{e^{2\pi n} - 1} - \cos 2\pi n x. \tag{1.5}$$

The spectral problem (1.2), (1.3) is not self-adjoint and it is easy to see that the following problem with the eigenvalues (1.4) will be adjoint to it

$$\sigma^{(IV)}(x) - \lambda\sigma(x) = 0, \quad 0 < x < 1, \tag{1.6}$$

$$\sigma(0) = \sigma(1), \quad \sigma'(1) = 0, \quad \sigma''(0) = \sigma''(1), \quad \sigma'''(0) = 0. \tag{1.7}$$

We also consider adjoint to it problem (1.6), (1.7). Solving this problem, it is not difficult to see that the eigenfunctions, corresponding to eigenvalues (1.4), have the form

$$\sigma_0(x) = 1, \quad \sigma_{1n}(x) = \frac{e^{2\pi nx} + e^{2\pi n(1-x)}}{e^{2\pi n} - 1} - \sin 2\pi nx, \quad \sigma_{2n}(x) = -2\cos 2\pi nx. \tag{1.8}$$

Lemma 1.1 ([34])**.** *Systems of functions* (1.5) *and* (1.8) *are biorthogonal systems in* $L_2[0,1]$*:*

$$\left(\vartheta_0, \sigma_0\right)_0 = 1, \quad \left(\vartheta_{ik}, \sigma_{jn}\right)_0 = \begin{cases} 1, & k = n, \ i = j \\ 0, & k \neq n, \ i \neq j \end{cases}, \quad i, j = 1, 2, \quad n, k = 1, 2, ...$$

Lemma 1.2 ([34])**.** *The systems of functions* (1.5) *and* (1.8) *are minimal in* $L_2[0,1]$.

Lemma 1.3 ([34])**.** *The system of functions* (1.5) *and* (1.8) *is complete in the space* $L_2[0,1]$.

Lemma 1.4 ([34])**.** *The system of functions* (1.5) *and* (1.8) *forms the Riesz basis in* $L_2[0,1]$.

2 Formal solution of the problem (0.1)-(0.3)

Taking into account the formulas (1.5) and (1.8) we look for a solution to the problem (0.1)–(0.3) in the form of following Fourier series:

$$U(t,x) = u_0(t)\,\vartheta_0(x) + \sum_{n=1}^{\infty}\Big(u_{1,n}(t)\,\vartheta_{1,n}(x) + u_{2,n}(t)\,\vartheta_{2,n}(x)\Big), \tag{2.1}$$

where

$$u_0(t) = \int_0^1 U(t,y)\sigma_0(y)dy, \quad u_{\kappa,n}(t) = \int_0^1 U(t,y)\sigma_{\kappa,n}(y)dy, \quad \kappa = 1, 2. \tag{2.2}$$

Similarly, the function $f(t,x)$ we write as

$$f(t,x) = f_0(t)\,\vartheta_0(x) + \sum_{n=1}^{\infty}\Big(f_{1,n}(t)\,\vartheta_{1,n}(x) + f_{2,n}(t)\,\vartheta_{2,n}(x)\Big), \tag{2.3}$$

where

$$f_0(t) = \int_0^1 f(t,y)\sigma_0(y)dy, \quad f_{\kappa,n}(t) = \int_0^1 f(t,y)\sigma_{\kappa,n}(y)dy, \quad \kappa = 1, 2. \tag{2.4}$$

Let the function $U(t,x)$ in (2.1) be a solution to boundary problem (0.1)–(0.3). Then, substituting representations (2.1), (2.3) into equation (0.1) and taking (2.2), (2.4) into account, we obtain

$$u_0'''(t)\vartheta_0(x) + \sum_{n=1}^{\infty}\Big(u_{1,n}'''(t)\,\vartheta_{1,n}(x) + u_{2,n}'''(t)\,\vartheta_{2,n}(x)\Big) +$$

$$+\sum_{n=1}^{\infty}\lambda_n\Big(u'''_{1,n}(t)\,\vartheta_{1,n}(x)+u'''_{2,n}(t)\vartheta_{2,n}(x)\Big)+\sum_{n=1}^{\infty}\omega^3\lambda_n\Big(u_{1,n}(t)\,\vartheta_{1,n}(x)+u_{2,n}(t)\vartheta_{2,n}(x)\Big)=$$

$$=f_0(t)\vartheta_0(x)+\sum_{n=1}^{\infty}\Big(f_{1,n}(t)\,\vartheta_{2,n}(x)+f_{2,n}(t)\,\vartheta_{2,n}(x)\Big).$$

Hence, we obtain two differential equations: scalar differential equation and a countable system of third order ordinary differential equations

$$u'''_0(t)=f_0(t), \tag{2.5}$$

$$u'''_{\kappa,n}(t)+\mu_n\omega^3 u_{\kappa,n}(t)=\frac{f_{\kappa,n}(t)}{1+\lambda_n}, \tag{2.6}$$

First, we solve the equation (2.5):

$$u_0(t)=C_{1,0}+tC_{2,0}+\frac{t^2}{2}C_{3,0}+\int_0^t\frac{(t-s)^2}{2}f_0(s)ds. \tag{2.7}$$

To find $C_{j,0}(j=1,2,3)$, we rewrite the conditions (0.3) as

$$u_0(0)=\varphi_{1,0},\quad u_0(T)=\varphi_{2,0},\quad u'_0(0)=\varphi_{3,0}, \tag{2.8}$$

where

$$\varphi_{j,0}=\int_0^1\varphi_j(y)\sigma_0(y)dy,\quad j=1,2,3.$$

By virtue of (2.8), from (2.7) we obtain that

$$C_{1,0}=\varphi_{1,0},\quad C_{2,0}=\varphi_{2,0},\quad C_{3,0}=\varphi_{3,0}-\frac{2}{T^2}\varphi_{1,0}-\frac{2}{T}\varphi_{2,0}-\frac{2}{T^2}\int_0^T\frac{(T-s)^2}{2}f_0(s)ds. \tag{2.9}$$

Substituting (2.9) into the representation (2.7), derived

$$u_0(t)=\varphi_{1,0}\left[1-\left(\frac{t}{T}\right)^2\right]+\varphi_{2,0}\left[t-\frac{t^2}{T}\right]+\left(\frac{t}{T}\right)^2\varphi_{3,0}+\int_0^T K_0(t,s)f_0(s)ds, \tag{2.10}$$

where

$$K_0(t,s)=\begin{cases}\dfrac{(t-s)^2}{2}-\dfrac{(T-s)^2}{T^2}, & 0\le s<t,\\ -\dfrac{(T-s)^2}{T^2}, & t\le s\le T.\end{cases}$$

Next, we will integrate the countable system of third-order differential equations (2.6). We consider roots of the characteristic equation $r^3+\mu_n\omega^3=0$ for the homogeneous differential equation $u'''_{\kappa,n}(t)+\mu_n\omega^3u_{\kappa,n}(t)=0$:

$$r_1=-\sqrt[3]{\mu_n}\omega,\quad r_{2/3}=\left(\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right)\sqrt[3]{\mu_n}\omega.$$

So, the general solution of the homogeneous equation presents as [28]–[30]

$$u_{\kappa,n}(t,\omega)=A_{\kappa,1,n}b_{1,n}(t,\omega)+A_{\kappa,2,n}b_{2,n}(t,\omega)+A_{\kappa,3,n}\,b_{3,n}(t,\omega),\quad \kappa=1,2,\;\; \tau_n(\omega)=\frac{\sqrt[3]{\mu_n}\omega}{2},$$

where $A_{\kappa,j,n}(j=1,2,3)$ are yet arbitrary coefficients, which will be determined from (2.8),

$$b_{1,n}(t,\omega)=e^{-2\tau_n(\omega)t},\quad b_{2,n}(t,\omega)=e^{\tau_n(\omega)t}\cos\sqrt{3}\tau_n(\omega)t,\quad b_{3,n}(t,\omega)=e^{\tau_n(\omega)t}\sin\sqrt{3}\tau_n(\omega)t.$$

The solution that satisfies the equation (2.6) with the boundary conditions (2.8) has the form:

$$u_{\kappa,n}(t,\omega)=A_{\kappa,1,n}(t,\omega)\,b_{1,n}(t,\omega)+A_{\kappa,2,n}(t,\omega)\,b_{2,n}(t,\omega)+A_{\kappa,3,n}(t,\omega)\,b_{3,n}(t,\omega)+$$

$$+B_{\kappa,1,n}b_{1,n}(t,\omega)+B_{\kappa,2,n}b_{2,n}(t,\omega)+B_{\kappa,3,n}b_{3,n}(t,\omega),\quad \kappa=1,2, \tag{2.11}$$

where

$$A_{\kappa,1,n}(t,\omega)=\frac{1}{12\tau_n^2(\omega)}\int\limits_0^t e^{2\tau_n(\omega)s}g_{\kappa,n}(s)ds,\quad \kappa=1,2, \tag{2.12}$$

$$A_{\kappa,2,n}(t,\omega)=-\frac{1}{6\tau_n^2(\omega)}\int\limits_0^t e^{-\tau_n(\omega)s}\cos\left(\sqrt{3}\tau_n(\omega)s-\frac{\pi}{3}\right)g_{\kappa,n}(s)ds, \tag{2.13}$$

$$A_{\kappa,3,n}(t,\omega)=-\frac{1}{6\tau_n^2(\omega)}\int\limits_0^t e^{-\tau_n(\omega)s}\sin\left(\sqrt{3}\tau_n(\omega)s-\frac{\pi}{3}\right)g_{\kappa,n}(s)ds. \tag{2.14}$$

Next, we find the following functions from (2.12)–(2.14) by differentiations

$$A'_{\kappa,1,n}(t,\omega)=\frac{e^{2\tau_n(\omega)t}}{12\tau_n^2(\omega)}g_{\kappa,n}(t),\quad A'_{\kappa,2,n}(t,\omega)=-\frac{e^{-\tau_n(\omega)t}\cos\left(\sqrt{3}\tau_n(\omega)t-\frac{\pi}{3}\right)}{6\tau_n^2(\omega)}g_{\kappa,n}(t), \tag{2.15}$$

$$A'_{\kappa,3,n}(t,\omega)=-\frac{e^{-\tau_n(\omega)t}\sin\left(\sqrt{3}\tau_n(\omega)t-\frac{\pi}{3}\right)}{6\tau_n^2(\omega)}g_{\kappa,n}(t). \tag{2.16}$$

To find the unknown (arbitrary) coefficients $B_{\kappa,i,n}$ $(i=1,2,3)$ in the presentation (2.11), we use the values of the functions (2.12)–(2.16) at the points $t=0,\quad t=T$. So, taking into account values (2.12)–(2.14), from (2.11) we derived

$$\varphi_{\kappa,1,n}(\omega)=u_{\kappa,n}(0,\omega)=A_{\kappa,1,n}(0,\omega)b_{1,n}(0,\omega)+A_{\kappa,2,n}(0,\omega)b_{2,n}(0,\omega)+$$

$$+A_{\kappa,3,n}(0,\omega)\,b_{3,n}(0,\omega)+B_{\kappa,1,n}(\omega)\,b_{1,n}(0,\omega)+B_{\kappa,2,n}(\omega)\,b_{2,n}(0,\omega)+B_{\kappa,3,n}(\omega)\,b_{3,n}(0,\omega)=$$

$$=0+0+0+B_{1,n}(\omega)+B_{2,n}(\omega)+0.$$

Hence, we have

$$B_{\kappa,1,n}(\omega)+B_{\kappa,2,n}(\omega)=\varphi_{\kappa,1,n}(\omega). \tag{2.17}$$

Similarly, we derive

$$\varphi_{\kappa,2,n}(\omega)=u_{\kappa,n}(T,\omega)=$$

$$=A_{\kappa,1,n}(T,\omega)b_{1,n}(T,\omega)+A_{\kappa,2,n}(T,\omega)\,b_{2,n}(T,\omega)+A_{\kappa,3,n}(T,\omega)\,b_{3,n}(T,\omega)+$$

$$+B_{\kappa,1,n}(\omega)\,b_{1,n}(T,\omega)+B_{\kappa,2,n}(\omega)\,b_{2,n}(T,\omega)+B_{\kappa,3,n}(\omega)\,b_{3,n}(T,\omega)=$$
$$=B_{\kappa,1,n}(\omega)\,b_{1,n}(T,\omega)+B_{\kappa,2,n}(\omega)\,b_{2,n}(T,\omega)+B_{\kappa,3,n}(\omega)\,b_{3,n}(T,\omega)+\alpha_{\kappa,n}(T,\omega),\quad \kappa=1,2,$$

where

$$\alpha_{\kappa,n}(T,\omega)=A_{\kappa,1,n}(T,\omega)\,b_{1,n}(T,\omega)+A_{\kappa,2,n}(T,\omega)\,b_{2,n}(T,\omega)+A_{\kappa,3,n}(T,\omega)\,b_{3,n}(T,\omega).$$

Hence, we obtain

$$b_{1,n}(T,\omega)\,B_{\kappa,1,n}(\omega)+b_{2,n}(T,\omega)\,B_{\kappa,2,n}(\omega)+b_{3,n}(T,\omega)\,B_{\kappa,3,n}(\omega)=$$
$$=\varphi_{\kappa,2,n}(\omega)-\alpha_{\kappa,n}(T,\omega),\quad \kappa=1,2. \tag{2.18}$$

We differentiate the function

$$u'_{\kappa,n}(t,\omega)=A'_{\kappa,1,n}(t,\omega)\,b_{1,n}(t,\omega)+A_{\kappa,1,n}(t,\omega)\,b'_{1,n}(t,\omega)+A'_{\kappa,2,n}(t,\omega)\,b_{2,n}(t,\omega)+$$
$$+A_{\kappa,2,n}(t,\omega)\,b'_{2,n}(t,\omega)+A'_{\kappa,3,n}(t,\omega)\,b_{3,n}(t,\omega)+A_{\kappa,3,n}(t,\omega)\,b'_{3,n}(t,\omega)+$$
$$+B_{\kappa,1,n}(\omega)\,b'_{1,n}(t,\omega)+B_{\kappa,2,n}(\omega)\,b'_{2,n}(t,\omega)+B_{\kappa,3,n}(\omega)\,b'_{3,n}(t,\omega).$$

Hence, by virtue of functions (2.15)–(2.16) at the point $t=0$ we derive

$$\varphi_{\kappa,3,n}(\omega)=u'_{\kappa,n}(0,\omega)=A'_{\kappa,1,n}(0,\omega)\,b_{1,n}(0,\omega)+A_{\kappa,1,n}(0,\omega)\,b'_{1,n}(0,\omega)+$$
$$+A'_{\kappa,2,n}(0,\omega)\,b_{2,n}(0,\omega)+A_{\kappa,2,n}(0,\omega)\,b'_{2,n}(0,\omega)+A'_{\kappa,3,n}(0,\omega)\,b_{3,n}(0,\omega)+$$
$$+A_{\kappa,3,n}(0,\omega)\,b'_{3,n}(0,\omega)+B_{\kappa,1,n}(\omega)\,b'_{1,n}(0,\omega)+B_{\kappa,2,n}(\omega)\,b'_{2,n}(0,\omega)+B_{\kappa,3,n}(\omega)\,b'_{3,n}(0,\omega).$$

From last relation, we obtain

$$-2\tau_n(\omega)\,B_{\kappa,1,n}(\omega)+\tau_n(\omega)\,B_{\kappa,2,n}(\omega)+\sqrt{3}\tau_n(\omega)\,B_{\kappa,3,n}(\omega)=\varphi_{\kappa,3,n}(\omega),\quad \kappa=1,2. \tag{2.19}$$

The algebraic equations (2.17)–(2.19) we write as a system of linear equations

$$\begin{cases} B_{\kappa,1,n}(\omega)+B_{\kappa,2,n}(\omega)=\varphi_{\kappa,1,n}(\omega),\\ b_{1,n}(T,\omega)\,B_{\kappa,1,n}(\omega)+b_{2,n}(T,\omega)\,B_{\kappa,2,n}(\omega)+b_{3,n}(T,\omega)\,B_{\kappa,3,n}(\omega)=\\ =\varphi_{\kappa,2,n}(\omega)-\alpha_{\kappa,n}(T,\omega),\\ -2\tau_n(\omega)\,B_{\kappa,1,n}(\omega)+\tau_n(\omega)\,B_{\kappa,2,n}(\omega)+\sqrt{3}\tau_n(\omega)\,B_{\kappa,3,n}(\omega)=\varphi_{\kappa,3,n}(\omega). \end{cases} \tag{2.20}$$

To solve the system (2.20) with respect to $B_{\kappa,i,n}(\omega)$ $(i=1;2;3)$, $\kappa=1,2$, we will calculate the main determinant of the system (2.20):

$$\Delta_n=\begin{vmatrix} 1 & 1 & 0\\ b_{1,n}(T,\omega) & b_{2,n}(T,\omega) & b_{3,n}(T,\omega)\\ -2\tau_n(\omega) & \tau_n(\omega) & \sqrt{3}\tau_n(\omega) \end{vmatrix}=$$
$$=\sqrt{3}\tau_n(\omega)\,b_{2,n}(T,\omega)-\tau_n(\omega)\,b_{3,n}(T,\omega)-\sqrt{3}\tau_n(\omega)\,b_{1,n}(T,\omega)-\tau_n(\omega)\,b_{3,n}(T,\omega)=$$
$$=-\sqrt{3}\tau_n(\omega)\,\left(e^{-2\tau_n(\omega)T}+2e^{\tau_n(\omega)T}\sin\left(\sqrt{3}\tau_n(\omega)T-\frac{\pi}{6}\right)\right),\quad \tau_n(\omega)=\frac{\sqrt[3]{\mu_n}\omega}{2}.$$

The value of the parameter ω, for which the equation $\Delta_n=0$ will have no solutions and $|\,\Delta_n\,|\geq\varepsilon_0>0$, we call as regular values of the parameter ω, and we will denote its set of solutions by Λ_0 and other values of parameter ω, we call as irregular values of the parameter: $\omega \in R \setminus (\Lambda_0 \cup \{0\})$.

For regular values $\omega \in \Lambda_0$ we will continue to solve the linear system (2.20) and obtain:

$$B_{\kappa,1,n}(\omega) = -\frac{2e^{\tau_n(\omega)T}\cos\left(\sqrt{3}\tau_n(\omega)T + \dfrac{\pi}{6}\right)}{\sqrt{3}\chi_{0,n}(T,\omega)}\varphi_{\kappa,1,n}(\omega) + \frac{1}{\chi_{0,n}(T,\omega)}\varphi_{\kappa,2,n}(\omega) +$$

$$+\frac{e^{\tau_n(\omega)T}\sin\left(\sqrt{3}\tau_n(\omega)T\right)}{\sqrt{3}\tau_n(\omega)\chi_{0,n}(T,\omega)}\varphi_{\kappa,3,n}(\omega) - \frac{1}{\chi_{0,n}(T,\omega)}\alpha_{\kappa,n}(T,\omega),$$

$$B_{\kappa,2,n}(\omega) = \frac{\sqrt{3}e^{-2\tau_n(\omega)T} + 2e^{\tau_n(\omega)T}\sin\left(\sqrt{3}\tau_n(\omega)T\right)}{\chi_{0,n}(T,\omega)}\varphi_{\kappa,1,n}(\omega) - \frac{1}{\chi_{0,n}(T,\omega)}\varphi_{\kappa,2,n}(\omega) +$$

$$+\frac{e^{\tau_n(\omega)T}\sin\left(\sqrt{3}\tau_n(\omega)T\right)}{\sqrt{3}\tau_n(\omega)\chi_{0,n}(T,\omega)}\varphi_{\kappa,3,n}(\omega) + \frac{1}{\chi_{0,n}(\omega)}\alpha_{\kappa,n}(T,\omega),$$

$$B_{\kappa,3,n}(\omega) = -\frac{e^{-2\tau_n(\omega)T} + 2e^{\tau_n(\omega)T}\cos\left(\sqrt{3}\tau_n(\omega)T\right)}{\sqrt{3}\chi_{0,n}(T,\omega)}\varphi_{\kappa,1,n}(\omega) + \frac{\sqrt{3}}{\chi_{0,n}(T,\omega)}\varphi_{\kappa,2,n}(\omega) +$$

$$+\frac{e^{-2\tau_n(\omega)T} - e^{\tau_n(\omega)T}\cos\left(\sqrt{3}\tau_n(\omega)T\right)}{\sqrt{3}\tau_n(\omega)\chi_{0,n}(T,\omega)}\varphi_{\kappa,3,n}(\omega) - \frac{\sqrt{3}}{\chi_{0,n}(T,\omega)}\alpha_{\kappa,n}(T,\omega),$$

where

$$\chi_{0,n}(T,\omega) = e^{-2\tau_n(\omega)T} + 2e^{\tau_n(\omega)T}\sin\left(\sqrt{3}\tau_n(\omega)T - \frac{\pi}{6}\right), \quad \tau_n(\omega) = \frac{\sqrt[3]{\mu_n}\omega}{2}.$$

Substituting the values of $B_{\kappa,j,n}(\omega)$ $(j = 1;2;3)$ into (2.11) and rearranging the sum with respect to $\varphi_{\kappa,j,n}(\omega)$ $(j = 1;2;3)$, we obtain the following:

$$B_{\kappa,1,n}(\omega)\, b_{1,n}(t,\omega) + B_{\kappa,2,n}(\omega)\, b_{2,n}(t,\omega) + B_{\kappa,3,n}(\omega)\, b_{3,n}(t,\omega) =$$

$$= \left[-\frac{2e^{\tau_n(\omega)(T-2t)}\cos\left(\sqrt{3}\tau_n(\omega)T + \dfrac{\pi}{6}\right)}{\sqrt{3}\chi_{0,n}(T,\omega)} + \right.$$

$$+\frac{\left(\sqrt{3}e^{\tau_n(\omega)(t-2T)} + 2e^{\tau_n(\omega)(T+t)}\sin\left(\sqrt{3}\tau_n(\omega)T\right)\right)\cos\left(\sqrt{3}\tau_n(\omega)t\right)}{\sqrt{3}\chi_{0,n}(T,\omega)} -$$

$$\left.-\frac{\left(e^{\tau_n(\omega)(t-2T)} + 2e^{\tau_n(\omega)(t+T)}\cos\left(\sqrt{3}\tau_n(\omega)T\right)\right)\sin\left(\sqrt{3}\tau_n(\omega)t\right)}{\sqrt{3}\chi_{0,n}(T,\omega)}\right]\varphi_{\kappa,1,n}(\omega) +$$

$$+\frac{e^{-2\tau_n(\omega)t} - e^{\tau_n(\omega)t}\left[\cos\left(\sqrt{3}\tau_n(\omega)t\right) - \sqrt{3}\sin\left(\sqrt{3}\tau_n(\omega)t\right)\right]}{\chi_{0,n}(T,\omega)}\varphi_{\kappa,2,n}(\omega) +$$

$$+\left[\frac{e^{\tau_n(\omega)(T-2t)}\sin\left(\sqrt{3}\tau_n(\omega)T\right) + e^{\tau_n(\omega)(T+t)}\sin\left(\sqrt{3}\tau_n(\omega)T\right)\cos\left(\sqrt{3}\tau_n(\omega)t\right)}{\sqrt{3}\tau_n(\omega)\chi_{0,n}(T,\omega)} + \right.$$

$$\left.+\frac{\left(e^{\tau_n(\omega)(t-2T)} - e^{\tau_n(\omega)(t+T)}\cos\left(\sqrt{3}\tau_n(\omega)T\right)\right)\sin\left(\sqrt{3}\tau_n(\omega)t\right)}{\sqrt{3}\tau_n(\omega)\,\chi_{0,n}(T,\omega)}\right]\varphi_{\kappa,3,n}(\omega) +$$

$$+\frac{-e^{-2\tau_n(\omega)t}+e^{\tau_n(\omega)t}\left[\cos\left(\sqrt{3}\tau_n(\omega)t\right)-\sqrt{3}\sin\left(\sqrt{3}\tau_n(\omega)t\right)\right]}{\chi_{0,n}(T,\omega)}\alpha_{\kappa,n}(T,\omega). \tag{2.21}$$

Taking (2.21) into account, we write the solution of the problem (0.1)–(0.3):

$$U_\kappa(t,x,\omega)=\sum_{n=1}^{\infty}u_{\kappa,n}(t,\omega)\,\vartheta_{\kappa,n}(x)=$$

$$=\sum_{n=1}^{\infty}\left[A_{\kappa,1,n}(t,\omega)\,b_{1,n}(t,\omega)+A_{\kappa,2,n}(t,\omega)\,b_{2,n}(t,\omega)+A_{\kappa,3,n}(t,\omega)\,b_{3,n}(t,\omega)+\right.$$

$$\left.+B_{\kappa,1,n}(\omega)\,b_{1,n}(t,\omega)+B_{\kappa,2,n}(\omega)\,b_{2,n}(t,\omega)+B_{\kappa,3,n}(\omega)\,b_{3,n}(t,\omega)\right]\vartheta_{\kappa,n}(x)=$$

$$=\sum_{n=1}^{\infty}\left[\chi_{1,n}(t,\omega)\,\varphi_{\kappa,1,n}(\omega)+\chi_{2,n}(t,\omega)\,\varphi_{\kappa,2,n}(\omega)+\chi_{3,n}(t,\omega)\,\varphi_{\kappa,3,n}(\omega)+\right.$$

$$\left.+\frac{1}{1+\lambda_n}\int_0^t K_{1,n}(t,s,\omega)\,f_{\kappa,n}(s,\omega)ds+\frac{1}{1+\lambda_n}\int_0^T \bar{K}_{1,n}(T,s,\omega)\,f_{\kappa,n}(s,\omega)ds\right]\vartheta_{\kappa,n}(x),$$

where

$$\chi_{1,n}(t,\omega)=\frac{2e^{\tau_n(\omega)(T+t)}\sin\left(\sqrt{3}\tau_n(\omega)(T-t)\right)}{\sqrt{3}\chi_{0,n}(T,\omega)},$$

$$\chi_{2,n}(t,\omega)=\frac{e^{-2\tau_n(\omega)t}-2e^{\tau_n(\omega)t}\cos\left(\sqrt{3}\tau_n(\omega)t+\dfrac{\pi}{6}\right)}{\chi_{0,n}(T,\omega)},\quad \chi_{3,n}(t,\omega)=$$

$$\frac{e^{\tau_n(\omega)(T-2t)}\sin\left(\sqrt{3}\tau_n(\omega)T\right)+e^{\tau_n(\omega)(t-2T)}\sin\left(\sqrt{3}\tau_n(\omega)t\right)+e^{\tau_n(\omega)(T+t)}\sin\left(\sqrt{3}\tau_n(\omega)(T-t)\right)}{\sqrt{3}\tau_n(\omega)\,\chi_{0,n}(T,\omega)},$$

$$K_{1,n}(t,s,\omega)=\frac{1}{12\tau_n^2(\omega)\chi_{0,n}(T,\omega)}e^{\tau_n(\omega)(t-s)}\left(e^{\tau_n(\omega)(t-s)}-2\cos\left(\sqrt{3}\tau_n(\omega)(t-s)+\frac{\pi}{3}\right)\right),$$

$$\bar{K}_{1,n}(T,s,\omega)=-\frac{\chi_{0,n}(t,\omega)}{12\tau_n^2(\omega)\chi_{0,n}(T,\omega)}\left(e^{2\tau_n(\omega)(T-s)}-2e^{\tau_n(\omega)(T-s)}\cos\left(\sqrt{3}\tau_n(\omega)(T-s)+\frac{\pi}{3}\right)\right).$$

Taking (2.10) into account, the last series we rewrite as

$$U(t,x,\omega)=2(1-x)\times$$

$$\times\left[\varphi_{1,0}\left[1-\left(\frac{t}{T}\right)^2\right]+\varphi_{2,0}\left[t-\frac{t^2}{T}\right]+\left(\frac{t}{T}\right)^2\varphi_{3,0}+\int_0^T K_0(t,s)f_0(s)ds\right]+$$

$$+\sum_{n=1}^{\infty}\sum_{\kappa=1}^{2}\left[\chi_{1,n}(t,\omega)\,\varphi_{\kappa,1,n}(\omega)+\chi_{2,n}(t,\omega)\,\varphi_{\kappa,2,n}(\omega)+\chi_{3,n}(t,\omega)\,\varphi_{\kappa,3,n}(\omega)+\right.$$

$$\left.+\frac{1}{1+\lambda_n}\int_0^T K_{2,n}(t,s,\omega)\,f_{\kappa,n}(s)ds\right]\vartheta_{\kappa,n}(x),\quad \kappa=1,2, \tag{2.22}$$

where

$$K_{2,n}(t,s,\omega)=\begin{cases}K_{1,n}(t,s,\omega), & t\le s\le T,\\ K_{1,n}(t,s,\omega)+\bar{K}_{1,n}(t,s,\omega), & 0\le s<t.\end{cases}$$

3 Unique solvability of the problem (0.1)-(0.3)

Theorem 3.1. *If there exists a solution of Problem* (0.1)–(0.3)*, then it is unique for regular values of the parameter* ω *from the set* Λ_0.

Proof. We consider the regular values of the parameter ω from the set Λ_0. Suppose that there exist two different solutions $U_1(t,x)$ and $U_2(t,x)$ to the problem (0.1)–(0.3). Then the difference $U(t,x) = U_1(t,x) - U_2(t,x)$ is a solution of the equation (0.1), satisfying the conditions (0.2) and (0.3) with functions $\varphi(x) \equiv 0$, $f(t,x) \equiv 0$. Then, it follows from formulas (2.7)–(2.9) and (2.21) that

$$u_{\kappa,n}(t) = \int\limits_0^1 U(t,y)\sigma_{\kappa,n}(y)dy \equiv 0, \quad \kappa = 1,2.$$

From this, due to the completeness of the system (1.5) and (1.8) in the space $L_2[0,1]$, it follows that $U(t,x) = 0$ almost everywhere on $[0,1]$ for any $t \in [0,T]$. Since $U(t,x) \in C(\bar{\Omega})$, it follows that $U(x,t) \equiv 0$ in $\bar{\Omega}$. The Theorem 3.1 is proved. □

Smoothness condition. Let in the domain $[0,1]$ the functions $\varphi_j(x)$ $(j = 1,2,3)$ and $f(t,x)$ satisfied the conditions

$$\varphi_j(x) \in C_x^5[0,1], \quad \varphi_j(1) = \frac{d^2}{dx^2}\varphi_j(0) = \frac{d^4}{dx^4}\varphi_j(1) = 0,$$

$$\frac{d}{dx}\varphi_j(0) = \frac{d}{dx}\varphi_j(1), \quad \frac{d^3}{dx^3}\varphi_j(0) = \frac{d^3}{dx^3}\varphi_j(1),$$

$$f(t,x) \in C_{t,x}^{0,2}([0,T]\times[0,1]), \quad f(t,1) = \frac{d^2}{dx^2}f(t,0) = \frac{d^4}{dx^4}f(t,1) = 0,$$

$$\frac{d}{dx}f(t,0) = \frac{d}{dx}f(t,1), \quad \frac{d^3}{dx^3}f(t,0) = \frac{d^3}{dx^3}f(t,1).$$

Then, we integrate by parts

$$\varphi_{\kappa,j,n} = \int\limits_0^1 \varphi_j(y)\sigma_\kappa(y)dy, \quad \kappa = 1,2, \quad j = 1,2,3$$

five times on the variable x, respectively, and obtain

$$\varphi_{1,j,n} = -\left(\frac{1}{2\pi}\right)^5 \frac{\varphi_{1,j,n}^{(V)}}{n^5}, \quad j = 1,2,3,$$

$$\varphi_{1,j,n}^{(V)} = \int\limits_0^1 \frac{\partial^5\varphi_j(y)}{\partial y^5}\left(\frac{e^{2\pi ny} - e^{2\pi n(1-y)}}{e^{2\pi n} - 1} + \cos 2\pi ny\right)dy,$$

$$\varphi_{2,j,n} = \left(\frac{1}{2\pi}\right)^5 \frac{\varphi_{2,j,n}^{(V)}}{n^5}, \quad \varphi_{2,j,n}^{(V)} = 2\int\limits_0^1 \frac{\partial^5\varphi_j(y)}{\partial y^5}\sin 2\pi ny dy.$$

Similarly, we have

$$f_{1,n}(t) = -\left(\frac{1}{2\pi}\right)^2 \frac{f''_{1,n}(t)}{n^2}, \quad f''_{1,n}(t) = \int\limits_0^1 \frac{\partial^2 f(t,y)}{\partial y^2} \left(\frac{e^{2\pi ny} - e^{2\pi n(1-y)}}{e^{2\pi n} - 1} + \cos 2\pi ny\right) dy,$$

$$f_{2,n}(t) = -\left(\frac{1}{2\pi}\right)^2 \frac{f''_{2,n}(t)}{n^2}, \quad f''_{2,n}(t) = \int\limits_0^1 \frac{\partial^2 f(t,y)}{\partial y^2} \sin 2\pi ny dy.$$

In addition, we have

$$\left\| \vec{\varphi}^{(V)}_{\kappa,j} \right\|_{\ell_2} \leq C_1 \left\| \frac{\partial^5 \varphi_j(x)}{\partial x^5} \right\|_{L_2[0,1]},$$

$$\left\| \max_{0\leq t\leq T} \vec{f}''_\kappa(t) \right\|_{\ell_2} \leq C_2 \left\| \max_{0\leq t\leq T} \frac{\partial^2 f(t,x)}{\partial x^2} \right\|_{L_2[0,1]}, \quad 0 < C_i = \text{const}, \quad \kappa = 1,2.$$

4 Convergence of the series

Theorem 4.1. *Let the smoothness conditions be satisfied. Then, for the regular values of the parameter ω from the set Λ_0 the function (2.22) will be belonged to the class $U(t,x) \in C(\overline{\Omega}) \cap C^{3,4}_{t,x}(\Omega)$.*

Proof. We consider the series (2.22) and

$$\frac{\partial^6}{\partial t^3 \partial x^3} U(t,x,\omega) =$$

$$+\sum_{n=1}^{\infty}\sum_{\kappa=1}^{2}\Big[\chi_{1,n}(t,\omega)\,\varphi_{\kappa,1,n}(\omega) + \chi_{2,n}(t,\omega)\,\varphi_{\kappa,2,n}(\omega) + \chi_{3,n}(t,\omega)\,\varphi_{\kappa,3,n}(\omega) +$$

$$+\frac{1}{1+\lambda_n}\int\limits_0^T K_{2,n}(t,s,\omega) f_{\kappa,n}(s) ds\Big]''' \vartheta'''_{\kappa,n}(x), \tag{4.1}$$

$$\frac{\partial^7}{\partial t^3 \partial x^4} U(t,x,\omega) =$$

$$+\sum_{n=1}^{\infty}\sum_{\kappa=1}^{2}\Big[\chi_{1,n}(t,\omega)\,\varphi_{\kappa,1,n}(\omega) + \chi_{2,n}(t,\omega)\,\varphi_{\kappa,2,n}(\omega) + \chi_{3,n}(t,\omega)\,\varphi_{\kappa,3,n}(\omega) +$$

$$+\frac{1}{1+\lambda_n}\int\limits_0^T K_{2,n}(t,s,\omega) f_{\kappa,n}(s) ds\Big]''' \vartheta^{(IV)}_{\kappa,n}(x). \tag{4.2}$$

The proofs of convergence of the series (4.1) and (4.2) are similar. So, we will prove of convergence for the following series

$$\sum_{n=1}^{\infty} n^4 \Big[\chi'''_{1,n}(t,\omega)\,\varphi_{1,1,n}(\omega) + \chi'''_{2,n}(t,\omega)\,\varphi_{1,2,n}(\omega) + \chi'''_{3,n}(t,\omega)\,\varphi_{1,3,n}(\omega) +$$

$$+\frac{1}{1+n^4}\int\limits_0^T K'''_{2,n}(t,s,\omega)f_{1,n}(s)ds\Bigg]\sin 2\pi nx, \tag{4.3}$$

$$\sum_{n=1}^{\infty} n^4\Big[\chi'''_{1,n}(t,\omega)\,\varphi_{2,1,n}(\omega)+\chi'''_{2,n}(t,\omega)\,\varphi_{2,2,n}(\omega)+\chi'''_{3,n}(t,\omega)\,\varphi_{2,3,n}(\omega)+$$

$$+\frac{1}{1+n^4}\int\limits_0^T K'''_{2,n}(t,s,\omega)f_{2,n}(s)ds\Bigg]\left(\frac{e^{2\pi nx}-e^{2\pi n(1-x)}}{e^{2\pi n}-1}-\cos 2\pi nx\right). \tag{4.4}$$

Applying the smoothness conditions and Bessel inequalities to (4.3), we have:

$$\max_{n=1,2,\dots}|\,\chi'''_{1,n}(t,\omega)\,|\sum_{n=1}^{\infty} n^4\,|\,\varphi_{1,1,n}(\omega)\,|+\max_{n=1,2,\dots}|\,\chi'''_{2,n}(t,\omega)\,|\sum_{n=1}^{\infty} n^4\,|\,\varphi_{1,2,n}(\omega)\,|+$$

$$+\max_{n=1,2,\dots}|\,\chi'''_{3,n}(t,\omega)\,|\sum_{n=1}^{\infty} n^4\,|\,\varphi_{1,3,n}(\omega)\,|+\sum_{n=1}^{\infty} n^4\frac{1}{1+n^4}\int\limits_0^T |\,K'''_{2,n}(t,s,\omega)\,|\,|\,f_{1,n}(s)\,|\,ds\le$$

$$\le\max_{n=1,2,\dots}\max_{0\le t\le T}|\,\chi'''_{1,n}(t,\omega)\,|\sum_{n=1}^{\infty} n^4\left|\frac{\varphi^{(V)}_{1,1,n}(\omega)}{n^5}\right|+\max_{n=1,2,\dots}\max_{0\le t\le T}|\,\chi'''_{2,n}(t,\omega)\,|\sum_{n=1}^{\infty} n^4\left|\frac{\varphi^{(V)}_{1,2,n}(\omega)}{n^5}\right|+$$

$$+\max_{n=1,2,\dots}\max_{0\le t\le T}|\,\chi'''_{3,n}(t,\omega)\,|\sum_{n=1}^{\infty} n^4\left|\frac{\varphi^{(V)}_{1,3,n}(\omega)}{n^5}\right|+\max_{n=1,2,\dots}\max_{0\le t\le T}\int\limits_0^T |\,K'''_{2,n}(t,s,\omega)\,|\,ds\sum_{n=1}^{\infty}|\,f_{1,n}(t)\,|\le$$

$$\le\sum_{j=1}^{3}\delta_j(\omega)\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}\sqrt{\sum_{n=1}^{\infty}\left|\,\varphi^{(V)}_{1,j,n}(\omega)\,\right|^2}+\delta_K(\omega)\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^4}}\sqrt{\sum_{n=1}^{\infty}\max_{0\le t\le T}\left|\,f''_{1,n}(t)\,\right|^2}\le$$

$$\le\frac{\pi^2}{6}\left[C_1\sum_{j=1}^{3}\delta_j(\omega)\left\|\frac{\partial^5\varphi_j(x)}{\partial x^5}\right\|_{L_2[0,1]}+C_2\delta_K(\omega)\left\|\max_{0\le t\le T}\frac{\partial^2 f(t,x)}{\partial x^2}\right\|_{L_2[0,1]}\right]<\infty,$$

where

$$\delta_j(\omega)=\max_{n=1,2,\dots}\max_{0\le t\le T}|\,\chi'''_{j,n}(t,\omega)\,|,\quad j=1,2,3;\quad \delta_K(\omega)=\max_{n=1,2,\dots}\max_{0\le t\le T}\int\limits_0^T|\,K'''_{2,n}(t,s,\omega)\,|\,ds.$$

Similarly, one can prove that the second series (4.4) is converges

$$\max_{n=1,2,\dots}|\,\chi'''_{1,n}(t,\omega)\,|\sum_{n=1}^{\infty} n^4\,|\,\varphi_{2,1,n}(\omega)\,|+\max_{n=1,2,\dots}|\,\chi'''_{2,n}(t,\omega)\,|\sum_{n=1}^{\infty} n^4\,|\,\varphi_{2,2,n}(\omega)\,|+$$

$$+\max_{n=1,2,\dots}|\,\chi'''_{3,n}(t,\omega)\,|\sum_{n=1}^{\infty} n^4\,|\,\varphi_{2,3,n}(\omega)\,|+\sum_{n=1}^{\infty}\int\limits_0^T|\,K'''_{2,n}(t,s,\omega)\,|\,|\,f_{2,n}(s)\,|\,ds<\infty.$$

The theorem 4.1 is proved. □

Conclusion

In the domain $\Omega = \{(t,x) \,|\, 0 < t < T,\ 0 < x < l\}$ the unique classical solvability of the boundary value problem (0.1)–(0.3) for a seventh-order partial differential equation is considered. The equation depend from two independent arguments. First argument is time argument and with respect to this argument is considered nonlocal conditions. Second argument is spatial and the equation with respect to this argument is differential equation of fourth order and considered with Samarskii–Ionkin conditions. Spectral problems and adjoint spectral problems are studied. The Fourier series method is used. The unique solvability of boundary value problem (0.1)–(0.3) is derived in the form of the Fourier series. It is proved the convergence of series.

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