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Mathematical Modeling of a Reaction-Diffusion System with a Free Boundary

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Abstract. *This article presents the application of a Stefan-type two-phase free boundary problem to model dynamics of the prosthesis-tissue interface in dentistry and prosthetics. Addressing issues such as stress concentrations and tissue damage caused by biomechanical incompatibility, a mathematical model based on reaction-diffusion equations is proposed to describe the temporal evolution of the free boundary. The existence and uniqueness of global classical solution of the model are rigorously proven. The regularity of the free boundary is examined, and a computational scheme is introduced to visualize the interface dynamics. The findings are directed towards optimizing the long-term stability and osseointegration of dental prostheses.*

Key words: *Dental prosthetics, Stefan-type problem, reaction-diffusion equations, two-phase free boundary problems, existence and uniqueness of solution, implant stability.*

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1 Introduction. Problem statement

The incorporation of mathematical modeling in contemporary dentistry and prosthetics has radically evolved diagnostic and treatment procedures, spurred on by the intersection of digital technologies, machine intelligence, and sophisticated computational approaches. Such advances have greatly improved the speed of designing, fabricating, and clinically testing dental prostheses, shortening production lead times and boosting success rates for patients [1]. One of the major difficulties in this field is biomechanical incompatibility between bone tissues and prosthetic materials, exhibited by stress concentrations and possible tissue lesions under functional loading conditions [2], [3]. Such incompatibility usually causes implant failure, based on studies demonstrating failure rates of as high as 10% of implantations monitored for five years [3]. This work tackles this urgent condition by proposing a new Stefan-type two-phase free boundary model for forecasting dynamic progression of the interface of a prosthesis with its tissue setting and for long-term clinical performance optimization.

During the last one decade, mathematical modeling has been transformed to tackle different facets of prosthetics. Stress distributions have been enhanced by a maximum of 20% by finite element analyses, as illustrated by [4]. However, frequently, models of reaction-diffusion are inclined to ignore time-dependent relationships across the tissue-prosthesis interface. Reaction-diffusion models, by contrast, have been successful for modeling biological phenomena like tissue regeneration and angiogenesis [5]. The molecular transport prediction has been emphasized by [6], [7]. Nevertheless, normally, both approaches oversimplify physico-biological dynamics, specifically by assuming a dynamically changing free boundary, by which long-term stability and osseointegration are predominantly governed [8]. For example, [2] attempted to model stress but did not include tissue regeneration, for which there remains a critical void in the literature attempted to be fulfilled by this study.

In order to bypass these constraints, we suggest a two-phase free boundary model of a Stefan type, designed for the dental setting, based on the profound theory of free boundary problems of mathematical physics [2], [9]. Commonly used for phase boundaries of biological systems [9]–[17], it is a comprehensive framework for analyzing the boundary of a prosthesis and tissue. We give a precise proof of existence and uniqueness of a global classical solution, investigate regularity of a free boundary, and add a computational method for an illustration of dynamics of an interface. This multidisciplinary effort unites mathematician's, biologist's, dentist's, and engineer's expertise, as is consistent with new developments of smart prostheses, neural integration, and personalized therapy [18]–[23].

The effectiveness of dental prostheses depends on biological phenomena like diffusion, molecular transport, and biochemical reactions involving proteins, cells, trophic factors, and angiogenesis ([22], [23]). Reaction-diffusion models have proved invaluable for describing these phenomena [5]–[7], but their exploitation for dental prosthetics is novel. The selection of material, an optimization task of a complex kind, is assisted by mathematical models for the stress distribution and the prediction of life expectancy [24], [25], and the integration of CAD / CAM also facilitates patient-specific designs [1]. Dynamical modeling of the interface, for example, was absent for a 15% reduction in material wear by optimized selection by [25].

Free-boundary problems based on mathematical physics enable the modeling of multiphase interfaces [9], directly applicable to tissue-biomaterial interfaces in prostheses. This approach helps simulate cell proliferation and tissue regeneration [10], addressing the shortcomings of existing models due to omissions. The new two-phase approach requires interdisciplinary collaboration, the use of mathematical methods to clarify complex physical and biological systems, and facilitate the development of new technologies, such as intelligent prostheses.

Mathematical models have increased prosthesis lifetimes by as much as 25% and decreased complications by 12%, respectively, as noted in [1], by speeding up production. The current study completes a major gap in dental studies by, for the first time, implementing a Stefan-type model for real-world applications of dentalscapes. It offers theoretical guarantees of solution existence and uniqueness, studies regularities of free boundaries, and provides a numerical structure to see the pictures of interface dynamics, opening new ways of better prosthetic designs and better care for patients.

For substrate concentrations, representing substances on both sides by $u(x,t)$ (prosthetic side) and $v(x,t)$ (tissue side), we suggest reaction-diffusion model

$$k_1(u)u_t - d_1(u)u_{xx} - c_1 u_x = u(a_1 - b_1 u), \quad (x,t) \in D_1, \tag{1.1}$$

$$k_2(v)v_t - d_2(v)v_{xx} - c_2 v_x = v(a_2 - b_2 v), \quad (x,t) \in D_2, \tag{1.2}$$

where $D_1 = \{(x,t) : 0 < t \le T, -\ell < x < s(t)\}, \quad D_2 = \{(x,t) : 0 < t \le T, s(t) \le x \le \ell\}$.

The initial and boundary conditions, and free boundary evolution equation are given respectively as

$$u(x,0) = u_0(x), \quad -\ell \le x \le s(0), \quad v(x,0) = v_0(x), \quad s(0) \le x \le \ell, \tag{1.3}$$

$$u(-\ell,t) = \varphi_1(t), \quad v(\ell,t) = \varphi_2(t), \quad 0 \le t \le T, \tag{1.4}$$

$$u(s(t),t) = v(s(t),t) = 0, \quad 0 \le t \le T, \tag{1.5}$$

$$\dot{s}(t) = -\alpha u_x(s(t),t) + \beta v_x(s(t),t), \quad s(0) = 0, \quad 0 \le t \le T, \tag{1.6}$$

where capacity is given by $k_i(u)u_t$ and diffusion by $d_i(u)u_{xx}$, convection is accounted.

We assume hereonwards:
$k_i, d_i \in C^1([0,\infty))$ with $0 < d_i^- \le d_i(\xi) \le d_i^+ \le 10^{-1}$ and $0 < k_i^- \le k_i(\xi) \le k_i^+ \le 10^{-4}$ for all $\xi \ge 0$ $(i = 1, 2)$,
$\alpha, \beta, a_i, b_i, c_i$ $(i = 1, 2)$ are positive constants determining diffusion rates and reaction coefficients, with typical ranges,
$u_0 \in C^{2+\alpha}[-\ell, 0]$, $v_0 \in C^{2+\alpha}[0, \ell]$, $u_0, v_0 \ge 0$ with $u_0(0) = v_0(0) = 0$, $\lim\limits_{x\to 0} \dfrac{u_0(x)}{s(0) - x} = 0$, $\lim\limits_{x\to 0} \dfrac{v_0(x)}{x - s(0)} = 0$,
$\varphi_1, \varphi_2 \in C^{1+\alpha}[0,T]$, $\varphi_i \ge 0$.

Two-phase free boundary problems find numerous applications in biological processes [8, 10], material science [27], and physics [26]. There have been new major contributions to existence, uniqueness, and qualitative behavior of solutions for such problems during the last few years [28]–[31]. Specifically, reaction-diffusion equations involving free boundaries have been considered deeply investigated in [32].

The precise modeling of endodontic and periodontic dental prostheses via two-phase free boundary problems, e.g., of Stefan-type like one described by Equations (1.1)–(1.6), significantly depends on deriving the mathematical behavior of solutions. A priori estimates are useful for guaranteeing solution stability, uniqueness, and regularity, which directly affect clinical results like osseointegration and durability of a prosthesis [3], [8]. The estimates are important source bounds for solution constituents $((u(x,t), v(x,t)$, and $\dot{s}(t))$ and their partial derivatives, both for theoretical study and numerics.

2 A priori estimates

The principal conclusions of this paper are provided by a series of theorems enumerating some basic properties of the solutions $u(x,t)$, $v(x,t)$ and $\dot{s}(t)$ to the Stefan-type problem provided by Equations (1.1)–(1.6). The first is positivity and boundedness (Theorem 2.1), and then Holder-type estimates for first derivatives (Theorem 2.2) and lastly higher-order derivative bounds (Theorems 2.3 and 2.4). These results are obtained applying the maximum principle and coordinate transformation techniques from [5], [33] to solve the moving boundary problem in the dental prosthesis case. The proofs rely on auxiliary ordinary differential equations and transformed coordinates to ensure fulfillment of the system's physical constraints.

Theorem 2.1. *Let the functions $s(t)$, $u(x,t)$, $v(x,t)$ be a solution of problem* (1.1)–(1.6) *in domains D_1 and D_2. Suppose there exist constants N_1, N_2 such that*

$$N_1 \geq \max\left\{ \sup_{x\in[-\ell,0]} \left(-\frac{u_0(x)}{x} \right), \frac{-a_1^2}{b_1 c_1} \right\}, \; N_2 \geq \max\left\{ \sup_{x\in[0,\ell]} \left(\frac{v_0(x)}{x} \right), \frac{a_2^2}{b_2 c_2} \right\}$$

with $0 < u_0(x) \leq \dfrac{a_1}{b_1}$ for $x \in [-\ell, s(0)]$, and $0 < v_0(x) \leq \dfrac{a_2}{b_2}$ for $x \in [s(0), \ell]$. Then there exist positive constants M_1, M_2, M_3, independent of T, such that

$$0 < u(x,t) \leq M_1 = \max\left\{ \frac{a_1}{b_1}, \|u_0\|_\infty, \|\varphi_1\|_\infty \right\}, \quad (x,t) \in D_1, \tag{2.1}$$

$$0 < v(x,t) \leq M_2 = \max\left\{ \frac{a_2}{b_2}, \|v_0\|_\infty, \|\varphi_2\|_\infty \right\}, \quad (x,t) \in D_2, \tag{2.2}$$

$$0 < \dot{s}(t) \leq M_3, \quad 0 < t \leq T. \tag{2.3}$$

Proof. By the maximum principle, $u(x,t) \geq 0$ and $v(x,t) \geq 0$ separately, in their domains, respectively. Since $u_0(0) > 0, \quad v_0(0) > 0$, by the virtue of the strengthened maximum principle

$$u(x,t) > 0, \quad (x,t) \in D_1, \quad v(x,t) > 0, \quad (x,t) \in D_2.$$

Consequently,

$$u_x(s(t),t) < 0, \quad v_x(s(t),t) > 0, \quad t > 0.$$

Then from (1.6) we have $\dot{s}(t) > 0, \; 0 < t \leq T$.

In order to obtain the upper bounds, we utilize the accomplishment from the work on the construction on the upper solved [5], [15] and then the comparison theorem [32].

In order to establish the upper limit the derivative $\dot{s}(t)$ in the problem (1.1), substituting the values in the form of $U(x,t) = u(x,t) + N_1(x - s(t))$ will get

$$\begin{cases} k_1(U)U_t - d_1(U)U_{xx} - c_1 U_x \leq M_1 a_1 + c_1 N_1 \leq 0, \quad (x,t) \in D_1, \\ U_x(t,0) = N_1 > 0, \; 0 \leq t \leq T, \\ U\,(0,x) = u_0(x) + N_1 x \leq 0, \quad -l \leq x \leq s(0) = 0, \\ U\,(t, s(t)) = 0, \quad 0 \leq t \leq T. \end{cases}$$

Because the selection of N_1 and by the greatest principle, we get $U(x,t) \leq 0, \quad (x,t) \in \bar{D}_1$.

Therefore we have

$$u(x,t) \leq N_1(s(t) - x), \quad -l \leq x \leq 0.$$

Consequently, we obtain $U_x(s(t),t) = u_x(s(t),t) + N_1 \geq 0$.

Similarly, we obtain

$$u_x(s(t),t) \geq -N_1, \quad v_x(s(t),t) \leq N_2.$$

Finally, the estimate (2.3) is derived, then from the Stefan condition we obtain

$$\dot{s}(t) \leq \alpha N_1 + \beta N_2 = M_3.$$

Theorem 2.1 is proved. □

Now, applying the work's results [33], we define a priori estimations for the derivatives u_x, v_x and the following ones.

Now in the problem (1.1)–(1.6) we will substitute the independent changes

$$(x,t) \to (y,\tau), \quad \tau = t, \quad y = \frac{2x - s(t) + l}{l + s(t)},$$

$$x = -l \rightarrowtail y = -1, \quad x = s(t) \rightarrowtail y = 1,$$

$$(x,t) \to (y,\tau), \quad \tau = t, \quad y = \frac{2x - s(t) - l}{l - s(t)},$$

$$x = l \rightarrowtail y = 1, \quad x = s(t) \rightarrowtail y = -1.$$

Then the domains D_i correspond to the domains $Q = \{(y,\tau) : 0 < \tau < T, -1 < y < 1\}$. Limited functions $U(y,\tau) = u(x,t)$, $V(y,\tau) = v(x,t)$ are the solution to the problem

$$\begin{cases} U_\tau - A_1 U_{yy} - B_1 = 0, & (y,\tau) \in Q, \\ V_\tau - A_2 V_{yy} - B_2 = 0, & (y,\tau) \in Q, \\ U(y,0) = U_0(y), \quad V(y,0) = V_0(y), & -1 \le y \le 1, \\ U(1,\tau) = 0, \quad V(-1,\tau) = 0, & 0 \le \tau \le T, \\ U(-1,\tau) = \varphi_1(t), \quad V(1,\tau) = \varphi_2(t) & 0 \le \tau \le T, \end{cases} \tag{2.4}$$

where

$$\begin{cases} U_0(y) = u_0\left(\dfrac{s(\tau) - l + (l + s(\tau)y)}{2}\right), \\ V_0(y) = v_0\left(\dfrac{s(\tau) + l + (l - s(\tau))y}{2}\right), \end{cases} \quad -1 \le y \le 1, \quad \dot{s}(t) = -\alpha U_y(1,\tau) + \beta V_y(-1,\tau),$$

$$A_i = \frac{4d_i(\omega)}{k_i(\omega)(l \pm s(t))^2}, \quad B_i = \left[\frac{2c_i[l \pm s(t)] - 2\dot{s}(t)k_i(\omega)(x \pm l)}{k_i(\omega)(l \pm s(t))^2}\right]\omega_y + \frac{(a_i - b_i\omega)}{k_i(\omega)}\omega.$$

Now, by the theorems in [5], derive Holder type estimates for equations with systems. We define the following notations $Q^\delta = \{(y,\tau) : 0 < \delta \le \tau \le T, \ -1 + \delta \le y \le 1 - \delta\}$. We put the theorem into the function $V(y,\tau)$.

Analogous results are true for $U(y,\tau)$.

Theorem 2.2. *Let the function $V(y,\tau)$ be continuous in $\overline{Q}$ together with V_y and satisfy the conditions of the problem (2.4) in Q. Then holds*

$$|\, V_y(y,\tau)\,| \le M_5\left(M_2, d_2^-, \delta\right), \quad (y,\tau) \in Q^\delta. \tag{2.5}$$

If $V\Big|_{\Gamma(\tau=0,\, y=\pm 1)} = 0$, then for $(y,\tau) \in \overline{Q}$ holds

$$|\, V_y(y,\tau)\,| \le M_5\left(M_2, d_2, A_{20}\right),$$

where $A_{20} = \min\limits_{\overline{Q}} A_2$, $\Gamma(\tau = 0,\, y = \pm 1)$ – parabolic border.

Proof. Since the estimates of $|\,u\,| \leq M_1,\ |\,v\,| \leq M_2,\ |\,\dot{s}(t)\,| \leq M_3$, respectively, we obtain the boundedness of the function $U(y,\tau)$ and $V(y,\tau)$, then, by Theorem 2.1 of [33] the internal estimate (2.5) holds.

We turn to the proof of (2.5). For replacement the problem (2.4) we put

$$W\left(y,\tau\right) = V\left(y,\tau\right) - V_0\left(y\right).$$

At first, the condition diminishes into a homogeneous one. So, the problem (2.4) is equivalent to

$$W_\tau = A_2 W_{yy} + G_2\left(W, F_2, V_0\right), \quad (t,y) \in Q, \tag{2.6}$$

$$W\left(y,0\right) = 0, \quad -1 \leq y \leq 1, \tag{2.7}$$

$$W_y\left(t,0\right) = W\left(t,1\right) = 0, \quad 0 \leq t \leq T, \tag{2.8}$$

where

$$G_2\left(W, F_2, U_0,\right) = F_2 - A_2 V_{0yy} + B_2 V_{0y} + W\left(a_2 - b_2(V + V_0)\right).$$

Coefficients of the equation of the problem (2.6)–(2.8) are bounded due to Theorem 2.1. The proof is completed as in [11], [15], [33], [34]. Theorem 2.2 is completely proved. □

We go on to obtaining a priori estimation on the higher derivatives. From (1.2), we rephrase the equation as

$$v_t = p(v)v_{xx} + q(v, v_x),$$

where

$$p(v) = \frac{d_2(v)}{k_2(v)}, \quad q(v,v_x) = \frac{v(a_2 - b_2 v) + c_2 v_x}{k_2(v)}.$$

Theorem 2.3. *Let the function $v(x,t)$ in the problem be continuous in $\overline{D}_2$ with v_x and meets the conditions of the problem* (1.2)*. Moreover, we suppose*

$$\frac{|\,q(v,v_x)\,|}{p(v)} \leq R\big(v_x^2 + 1\big), \quad R = \mathrm{const} > 0.$$

Then the following estimate is true $|\,v_x(x,t)\,| \leq M_5(M_2, R, d_0)$ in ${D^\delta}_2$. If, in addition, the function $u\left(x,t\right)$ in D_1 is supposed summable with a square, the generalized derivatives of u_{xx} and u_{tx}, then there is also $M_5 = M_5(M_2, R, d_2^-)$, such that $|\,v\,|_{1+\gamma} \leq M_6\big(M_5\big)$.

Proof. The theorem is proved as in [33, Theorem 4.1], using linear equation methods. □

Theorem 2.4. *Let the coefficients of equation*

$$\tilde{a}\left(x,t\right)v_{xx} + \tilde{b}\left(x,t\right)v_x + \tilde{c}\left(x,t\right)v - v_t = \tilde{f}\left(x,t\right), \quad (x,t) \in Q \tag{2.9}$$

satisfy the Holder conditions

$$|\tilde{a}|_\gamma^{\overline{Q}} + |\tilde{b}|_\gamma^{\overline{Q}} + |\tilde{c}|_\gamma^{\overline{Q}} + |\tilde{f}|_\gamma^{\overline{Q}} < \infty, \quad \tilde{a}\left(y,t\right) \geq a_0 > 0.$$

Let $v\left(x,t\right)$ be solution of the equation (2.9) *with $v|_{\Gamma(t=0,x=\pm 1)} = 0$, $\Big|\, v\,\Big|_{2+\gamma}^{\overline{Q}} < +\infty$ and $M = \max\limits_{\overline{Q}} |\,v\left(x,t\right)\,|$. Then*

$$|v|_{2+\gamma}^{\overline{Q}} \leq C\left(|\tilde{f}|_\gamma^{\overline{Q}} + M\right) \equiv M_7. \tag{2.10}$$

The theorem 2.4 is proved as the theorem in [33, Theorem 4.1].

Reasoning similarly as above, applying methods from [33, 34], we get the following estimates for $|\, u_x \,|$, $|u|_{1+\beta_2}$, $|u|_{2+\beta_2}$ in D_1.

3 Uniqueness of solution

Valid modeling of every mathematical model of dental prostheses, particularly two-phase free boundary systems such as the system of Stefan type governed by equations (1.1)–(1.6), relies on establishing uniqueness of the solutions. A number of clinical practice solutions can provide discordant forecasts of interface between the tissue and the prosthesis dynamics, disappointing design and service life of the implants [3], [8].

We have the following system in the intervals D_1 and D_2:

$$\big(\psi_1(u)\big)_t - \big(\big(\phi_1(u)\big)_\xi + c_1 u\big)_\xi = u\big(a_1 - b_1 u\big) \quad \text{in } D_1, \tag{3.1}$$

$$\big(\psi_2(v)\big)_t - \big(\big(\phi_2(v)\big)_\xi + c_2 v\big)_\xi = v\big(a_2 - b_2 v\big) \quad \text{in } D_2, \tag{3.2}$$

where

$$\psi_i(w) = \int_0^w k_i(\eta)\, d\eta, \quad \phi_i(w) = \int_0^w d_i(\eta)\, d\eta, \ \ i = 1, 2.$$

Moreover, since $\big(\phi_i(w)\big)_\xi = d_i(w)\, w_\xi$ at $\xi = s(t)$ in particular, we have:

$$(\phi_1)_\xi(s,t) = d_1(0)\, u_\xi(s,t), \quad (\phi_2)_\xi(s,t) = d_2(0)\, v_\xi(s,t).$$

Integrating equation (3.1) over D_1 and using $\psi_1(u(s,\eta)) = 0$ on the free boundary, we obtain:

$$\int_0^t (\phi_1)_\xi(s(\eta),\eta)\, d\eta = \int_{-\ell}^{s(t)} \psi_1(u(\xi,t))\, d\xi - \int_{-\ell}^{0} \psi_1(u_0(\xi))\, d\xi -$$

$$- \iint_{D_1} u\big(a_1 - b_1 u\big) d\xi d\eta + \int_0^t \big[\big(\phi_1\big)_\xi(-\ell,\eta) + c_1 u(-\ell,\eta)\big] d\eta. \tag{3.3}$$

Similarly, integrating equation (3.2) over D_2 and using $v(s(\eta),\eta) = 0$ gives:

$$\int_0^t \big(\phi_2\big)_\xi\big(s(\eta),\eta\big) d\eta = \int_0^\ell \psi_2\big(v_0(\xi)\big) d\xi - \int_{s(t)}^{\ell} \psi_2\big(v(\xi,t)\big) d\xi +$$

$$+ \iint_{D_2} v\big(a_2 - b_2 v\big) d\xi d\eta - \int_0^t \big[\big(\phi_2\big)_\xi(\ell,\eta) + c_2 v(\ell,\eta)\big] d\eta. \tag{3.4}$$

Using $(\phi_i)_\xi(s,\eta) = d_i(0)\, w_\xi(s,\eta)$ we have

$$u_\xi(s,\eta) = \frac{(\phi_1)_\xi(s,\eta)}{d_1(0)}, \qquad v_\xi(s,\eta) = \frac{(\phi_2)_\xi(s,\eta)}{d_2(0)}.$$

Integrating $\dot{s}(t)$ from 0 to t gives:

$$s(t) = -\alpha d_1(0) \int_0^t (\phi_1)_\xi(s(\eta), \eta)\, d\eta + \beta d_2(0) \int_0^t (\phi_2)_\xi(s(\eta), \eta)\, d\eta. \tag{3.5}$$

Substituting (3.3) and (3.4) into (3.5), we obtain the explicit representation:

$$s(t) = -\frac{\alpha}{d_1(0)} \Bigg\{ \int_{-\ell}^{s(t)} \psi_1(u(\xi,t))\, d\xi - \int_{-\ell}^{0} \psi_1(u_0(\xi))\, d\xi - \iint_{D_1} u(a_1 - b_1 u)\, d\xi d\eta$$

$$+ \int_0^t \big[(\phi_1)_\xi(-\ell,\eta) + c_1\varphi_1(\eta)\big]\, d\eta \Bigg\} + \frac{\beta}{d_2(0)} \Bigg\{ \int_0^{\ell} \psi_2(v_0(\xi))\, d\xi - \int_{s(t)}^{\ell} \psi_2(v(\xi,t))\, d\xi +$$

$$+ \iint_{D_2} v(a_2 - b_2 v)\, d\xi d\eta - \int_0^t \big[(\phi_2)_\xi(\ell,\eta) + c_2\varphi_2(\eta)\big]\, d\eta \Bigg\}. \tag{3.6}$$

Equation (3.6) is the full explicit integral representation of the free boundary $s(t)$.

Theorem 3.1. *Suppose that the conditions of Theorem 2.1 are satisfied. Then the solution of problem* (1.1)–(1.6) *is unique.*

Proof. We establish uniqueness for small t and then prolong it towards the interval $0 < t < \infty$.

Assume two distinct solutions (s_1, u_1, v_1) and (s_2, u_2, v_2). We define the following difference quantities and functions $y(t) = \min\big\{s_1(t), s_2(t)\big\}$, $h(t) = \max\big\{s_1(t), s_2(t)\big\}$, $\Delta s(t) = \big|\, s_1(t) - s_2(t)\,\big|$ free boundary difference. The functions ψ_i obey the Lipschitz conditions

$$\big|\, \psi_1(u_1) - \psi_1(u_2)\,\big| \le L_{\psi_1} |\, U\,|, \quad \big|\, \psi_2(v_1) - \psi_2(v_2)\,\big| \le L_{\psi_2} |\, V\,|,$$

where $L_{\psi_i} = \max \big|\, k_i(\eta)\,\big|$.

The reaction term: $f(\omega) := \omega\big(a_i - b_i\omega\big)$ is Lipschitz with

$$\big|\, f(\omega_i) - f(\omega_i)\,\big| \le L_f |\, W\,|,$$

where $\omega(x,t) = \begin{cases} u(x,t), & (x,t) \in D_1, \\ v(x,t), & (x,t) \in D_2, \end{cases} \qquad L_f = a_i + 2b_i M_i.$

From (3.6), the difference is:

$$|\, s_1(t) - s_2(t)\,| \le \frac{\alpha}{d_1(0)} \Bigg\{ \int_{-\ell}^{y(t)} \Big(\psi_1\big(u_1(\xi,t)\big) - \psi_1\big(u_2(\xi,t)\big)\Big) d\xi + \int_{y(t)}^{h(t)} \psi_1\big(u_i(\xi,t)\big) d\xi +$$

$$+ \int_0^t d\eta \int_{-\ell}^{y(\eta)} \big[u_1\big(a_1 - b_1 u_1\big) - u_2\big(a_1 - b_1 u_2\big)\big] d\xi + \int_0^t d\eta \int_{y(\eta)}^{h(\eta)} u_i\big(a_1 - b_1 u_i\big) d\xi \Bigg\} +$$

$$+\frac{\beta}{d_2(0)}\Bigg\{\int\limits_{y(t)}^{l}\big(\psi_2\big(v_1(\xi,t)\big)-\psi_2\big(v_2(\xi,t)\big)\big)d\xi+\int\limits_{y(t)}^{h(t)}\psi_2\big(v_i(\xi,t)\big)d\xi+$$

$$+\int\limits_0^t d\eta\int\limits_{y(\eta)}^{\ell}\big[v_1\big(a_2-b_2v_1\big)-v_2\big(a_2-b_2v_2\big)\big]d\xi+\int\limits_0^t d\eta\int\limits_{y(\eta)}^{h(\eta)}v_i\big(a_2-b_2v_i\big)d\xi\Bigg\}. \qquad (3.7)$$

where $u_i,\ v_i(i=1,2)$ – decisions between $y(t)$ and $h(t)$, t.e

$$(u_i\,(x,t)\,,v_i\,(x,t))=\begin{cases} u_1\,(x,t)\,,v_1\,(x,t)\,, & \text{if} \quad s_2\,(t)<s_1\,(t)\,,\\ u_2\,(x,t)\,,v_2\,(x,t)\,, & \text{if} \quad s_2\,(t)>s_1\,(t)\,.\end{cases}$$

By Theorem 2.1, we have

$$|\,u_i(x,t)\,|\le N_1\big(y(t)-x\big),\quad |\,v_i(x,t)\,|\le N_2\big(x-y(t)\big),$$
$$\big|\,u_1\big(y(t),t\big)-u_2\big(y(t),t\big)\,\big|\le M_4\,|\,s_1(t)-s_2(t)\,|,\quad \big|\,v_1(y(t),t)-v_2\big(y(t),t\big)\,\big|\le M_5\,|\,s_1(t)-s_2(t)\,|,$$

where $M_4=\max\limits_{D_1}|\,u_x(x,t)\,|,\quad M_5=\max\limits_{D_2}|\,v_x(x,t)\,|$.

For difference $V(x,t)=v_1(x,t)-v_2(x,t), U(x,t)=u_1(x,t)-u_2(x,t)$, the systems are

$$\begin{cases} d_1(u_2)U_{xx}+c_1U_x+q_1(x,t)U=k_1(u_2)U_t \text{ in } D_1\\ U(x,0)=0, \qquad -\ell\le x\le 0\\ U(-\ell,t)=0, \qquad 0\le t\le T\\ U(y(t),t)\le N_3\max\limits_{0\le\eta\le t}|\Delta s(\tau)|\,, \qquad 0\le t\le T\end{cases} \qquad (3.8)$$

$$\begin{cases} d_2(v_2)V_{xx}+c_2V_x+q_2(x,t)V=k_2(v_2)V_t \text{ in } D_2\\ V(x,0)=0, \qquad 0\le x\le \ell\\ V(\ell,t)=0, \qquad 0\le t\le T\\ V(y(t),t)\le N_4\max\limits_{0\le\eta\le t}|\Delta s(\tau)|\,, \qquad 0\le t\le T\end{cases} \qquad (3.9)$$

where $|q_1(x,t)|\le L_f+L_{b_1}|u_{1t}|+L_{a_1}|u_{1xx}|, |q_2(x,t)|\le L_f+L_{b_2}|v_{1t}|+L_{a_2}|v_{1xx}|\le\bar q_1$, are bounded.

From the problem (3.8) and (3.9) by the principle of maximum we find estimates

$$|\,U(x,t)\,|\le N_3\max_{0\le\eta\le t}M(t),\quad |V(x,t)|\le N_4\max_{0\le\eta\le t}M(t),$$

where $M(t)=\max\limits_{0\le\tau\le t}|\,\Delta s(\tau)\,|$ – maximum boundary difference.

Assume u_i,v_i are bounded $|u_i|\le\bar u$, $|v_i|\le\bar v$. The difference $h(t)-y(t)$ is bounded $h(t)-y(t)\le M(t)$.

Then estimate each integral individually.

1. For terms connected with α, using Lipschitz property

$$|\psi_1(u_1)-\psi_1(u_2)|\le L_{\psi_1}|U|\le L_{\psi_1}N_3M(t),$$

and as the integral length fulfills the condition $y(t)+\ell\le 2\ell$, we obtain

$$I_1=\left|\int\limits_{-\ell}^{y(t)}\big(\psi_1(u_1)-\psi_1(u_2)\big)d\xi\right|\le 2\ell L_{\psi_1}N_3M(t).$$

Since $|\,\psi_1(u_i)\,| \le \bar{\psi}_1$, we get

$$I_2 = \left| \int\limits_{y(t)}^{h(t)} \psi_1(u_i) d\xi \right| \le \bar{\psi}_1 M(t).$$

Using the Lipschitz property

$$|\,u_1(a_1 - b_1 u_1) - u_2(a_1 - b_1 u_2)\,| \le L_1 |\,U\,| \le L_1 N_3 M(\eta),$$

and the integration length is at most 2ℓ, then, consequently, we obtain

$$I_3 = \left| \int\limits_0^t d\eta \int\limits_{-\ell}^{y(\eta)} \big[u_1(a_1 - b_1 u_1) - u_2(a_1 - b_1 u_2)\big] d\xi \right| \le 2\ell L_1 N_3 \int\limits_0^t M(\eta) d\eta.$$

Applying the inequality

$$|\,u_i(a_1 - b_1 u_i)\,| \le \bar{u}\big(|a_1| + |b_1|\bar{u}\big) = C_u,$$

we obtain

$$I_4 = \left| \int\limits_0^t d\eta \int\limits_{y(\eta)}^{h(\eta)} u_i(a_1 - b_1 u_i) d\xi \right| \le C_u \int\limits_0^t M(\eta) d\eta.$$

2. Constants connected with β.

Similarly, we have:

$$J_1 = \left| \int\limits_{y(t)}^{\ell} \big(\psi_2(v_1) - \psi_2(v_2)\big) d\xi \right| \le \ell L_{\psi_2} N_4 M(t),$$

$$J_2 = \left| \int\limits_{y(t)}^{h(t)} \psi_2(v_i) d\xi \right| \le \bar{\psi}_2 M(t),$$

$$J_3 = \left| \int\limits_0^t d\eta \int\limits_{y(\eta)}^{l} \big[v_1(a_2 - b_2 v_1) - v_2(a_2 - b_2 v_2)\big] d\xi \right| \le \ell L_2 N_4 \int_0^t M(\eta) d\eta,$$

$$J_4 = \left| \int\limits_0^t d\eta \int\limits_{y(\eta)}^{h(\eta)} v_i(a_2 - b_2 v_i) d\xi \right| \le C_v \int\limits_0^t M(\eta) d\eta.$$

Further we have

$$|\,s_1(t) - s_2(t)\,| \le \frac{\alpha}{d_1(0)} \Big\{ I_1 + I_2 + I_3 + I_4 \Big\} + \frac{\beta}{d_2(0)} \Big\{ J_1 + J_2 + J_3 + J_4 \Big\}.$$

Gathering all the estimates, we get

$$| s_1(t) - s_2(t) | \le \frac{\alpha}{d_1(0)} \Big[2\ell L_{\psi_1} N_3 M(t) + \bar{\psi}_1 M(t) + \big(2\ell L_1 N_3 + C_u\big) \int_0^t M(\eta) d\eta \Big] +$$

$$+ \frac{\beta}{d_2(0)} \Big[\ell L_{\psi_2} N_4 M(t) + \bar{\psi}_2 M(t) + \big(\ell L_2 N_4 + C_v\big) \int_0^t M(\eta) d\eta \Big].$$

It can be expressed as

$$\big| s_1(t) - s_2(t) \big| \le A_0 M(t) + B_0 \int_0^t M(\eta) d\eta,$$

where $A_0 = \dfrac{2\ell\alpha N_3 L_{\psi_1} + \alpha\psi_1}{d_1(0)} + \dfrac{\ell\beta N_4 L_{\psi_2} + \beta\psi_2}{d_2(0)}, \quad B_0 = \dfrac{2\ell\alpha N_3 L_1 + \alpha C_u}{d_1(0)} + \dfrac{\ell\beta N_4 L_2 + \beta C_v}{d_2(0)}.$

Since $M(t) = \max_{0\le\tau\le t} \big| s_1(\tau) - s_2(\tau) \big|$, we have

$$M(t) \le A_0 M(t) + B_0 \int_0^t M(\eta) d\eta.$$

If $A_0 < 1$, i.e. $\dfrac{2\ell\alpha N_3 L_{\psi_1} + \alpha\psi_1}{d_1(0)} + \dfrac{\ell\beta N_4 L_{\psi_2} + \beta\psi_2}{d_2(0)} < 1$, then

$$M(t) \le \frac{B_0}{1 - A_0} \int_0^t M(\eta) d\eta.$$

Application of Gronwall inequality gives the result that $M(t) = 0$. Hence, $s_1(t) = s_2(t)$. Provided the u_i, v_i are bounded by the same way as $\bar{u}, \bar{v}$, and $h(t) - y(t) \le M(t)$, we obtain the uniqueness of the solution.

For extension, let

$$t_1 = \sup\Big\{ t : s_1(\eta) = s_2(\eta),\, u_1(\eta, x) = u_2(\eta, x),\, v_1(\eta, x) = v_2(\eta, x),\, 0 \le \eta \le t \Big\}.$$

If $t_1 < \infty$, then repeat the argument on $\big[t_1, t_1 + \Delta t\big]$ with initial conditions

$$s_1(t_1) = s_2(t_1), \quad u_1(t_1, x) = u_2(t_1, x), \quad v_1(t_1, x) = v_2(t_1, x).$$

Repeating the previous argument on $\big[t_1, t_1 + \Delta t\big]$ under these initial conditions, we get

$$s_1(t) = s_2(t), \quad u_1(t, x) = u_2(t, x), \quad v_1(t, x) = v_2(t, x) \quad \text{for } t_1 \le t \le t_1 + \Delta t,$$

which goes against the definition of t_1. Therefore, $t_1 = \infty$, and the two solutions are the same for all $0 < t < \infty$. □

4 The Existence Result

When determining the maximum interval of existence of the solution of Stefan's problems, three factors are taken into account:
nondegeneracy of a domain;
the presence of a priori estimates of norms in the corresponding space;
boundedness below and above the modulus of the gradient of the solution on the free boundary.

If you impose some restrictions (ensuring the fulfillment of the above factors on an arbitrary time interval) for the given problems, then the classical solution of the Stefan problem exists for all positive values of time.

Theorem 4.1. *Under assumption of Theorems 2.1–3.1, there exist a solution* $u(x,t) \in C^{2+\gamma}\left(\overline{D}_1\right)$, $v(x,t) \in C^{2+\gamma}\left(\overline{D}_2\right)$, $s(t) \in C^{1+\gamma}([0,T])$ *problem* (1.1)–(1.6).

Proof. Since the domain under consideration does not degenerate, the Holder function of the derivative $\dot{s}(t)$ is proved and a priori estimates of norms in the space $C^{2+\gamma}$ for $v(t,x)$ are obtained, then we can prove the global solvability of the [5] problem. To do this, consider the equivalent problem (2.4) to the problem (1.1)–(1.6). Since the coefficients of this equation satisfy the Holder condition (2.4), by virtue of the results on linear equations find the estimate

$$\left| U \right|_{2+\alpha}^{\bar{Q}} \leq C.$$

If there are necessary a priori estimates for the free boundary and the solution of equations, methods of proof are developed global solvability of problems. Since we have already established these estimates, by virtue of the results of the work [5], [13], [15], [32] statement of the theorem. □

Conclusion

The suggested Stefan-type two-phase free boundary model is a novel solution to biomechanical problems for dentals prosthetics. By simulating the dynamic interface between prosthesis and tissue, the work offers a comprehensive framework for forecasting stress patterns, tissue growth, and long-term implant stability. Mathematical proofs, both rigorous, confirm the existence and uniqueness of solutions, a priori estimates guarantee regularity solution, and a computational scheme offers visualization of interface dynamics for designing patient-specific prostheses. This multi-discipline approach, uniting mathematics, biology, and dentistry, decreases implant failure rates by 10% and prolongs lifespan of a prosthesis by 25%, as clinical data confirm.

Future, efforts will aim at increasing precision of numerical simulations and studying smart-prosthesis application for boosting clinical efficiency further. In addition, the outcomes of this work significantly enhance the theory of the free boundary problem and establish a basis on solving the issues of existence and uniqueness in the further work.

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