Journal of Osh University "Differential Equations", 2026, Volume 1, Issue 1, P. 15–23.

UDC 517.927.6

Boundary Value Problem for a System of Fredholm Integro-Differential Equations with Maxima

A. Molybaikyzy*
Kazakh National Women's Pedagogical University, Almaty, Kazakhstan
E-mail: altynaimolybai@gmail.com, https://orcid.org/0009-0008-2452-5932

Akmaral D. Sarman
Aktobe Regional Zhubanov University, Department of Mathematics, Aktobe, Kazakhstan
E-mail: sadrasul8@gmail.com, https://orcid.org/0009-0006-3617-4071

Aigerim K. Tankeyeva
Aktobe Regional Zhubanov University, Department of Mathematics, Aktobe, Kazakhstan
E-mail: aigerimtankeyeva@gmail.com, https://orcid.org/0000-0002-3897-5909

Received: November 14, 2025; **First on-line published**: January 10, 2026

Abstract. *A Dzhumabayev's parameterization method is proposed to solve a "linear" two-point boundary value problem for a Fredholm integro-differential equation with maxima. The system of differential equations is changed with the system of integral equations. Then by the method of contracting mapping is studied the integral equation in the space $BD\big([0,T],\mathbb{R}^n\big)$. As a practical way of solving the original problem, it is transformed into a multipoint boundary value problem with parameters. Introduction of additional parameters yields a special Cauchy problem for a system of integro-differential equations with parameters on the subintervals. Using the solution to this problem, the boundary condition and continuity conditions of solutions at the interior points of the partition, we construct a system of linear algebraic equations in parameters. We give the algorithms of how to calculate the solutions of multipoint boundary value problem.*

Key words: *Fredholm integro-differential equation, parameterization method, boundary value problem, maxima, solvability.*

MSC 2020: 34A34, 34A45, 34B10.

1 Introduction. Problem statement

The theory of integro-differential equations are associated with its applications [1]–[4]. Various kind of problems for these equations have been studied and many works are published (see, for example, [5]–[15]).

In the present paper, on the interval $\Omega \equiv [0,T]$ we consider the "linear" boundary value problem

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)\max\Big\{x(\tau) : \tau \in \big[\delta_1(t);\delta_2(t)\big]\Big\}+$$

$$+\int_0^T K(t,s)x(s)ds + f(t), \quad t \in (0,T), \tag{1.1}$$

$$B_0 x(0) + C_0 x(T) = D_0, \;\; D_0 \in \mathbb{R}^n, \tag{1.2}$$

where $\delta_\kappa(t) \in C\big[\delta_0, \delta_1\big]$, $0 < \delta_0 = \max\limits_{\kappa=1,2} \min\limits_{0 \le t \le T} \delta_\kappa(t)$, $\delta_0 < \delta_1 = \max\limits_{\kappa=1,2} \max\limits_{0 \le t \le T} \delta_\kappa(t) < T$, $\kappa = 1,2$, the $(n \times n)$ matrices $A(t)$ and $B(t)$ are continuous functions on $[0,T]$ and $K(t,s)$ in $[0,T] \times [0,T]$ is continuous matrix-function, the $n-$dimensional vector-function $f(t)$ is also continuous on $[0,T]$, $\|x(t)\| = \max\limits_{i=1,N} |x_i(t)|$.

Denote by $C([0,T], \mathbb{R}^n)$ the space of all continuous functions $x : [0,T] \to \mathbb{R}^n$ with the norm $\|x(t)\|_1 = \max\limits_{t \in [0,T]} \|x(t)\|$. By a solution to problem (1.1), (1.2) we mean a continuously differentiable on $(0,T)$ vector-function $x(t) \in C([0,T], \mathbb{R}^n)$ satisfying equation (1.1) and boundary condition (1.2).

In [16]–[20], the problem (1.1),(1.2) is studied by the parameterization method. The interval $[0,T]$ was divided into N parts, the values of a solution to problem (1.1),(1.2) at the left endpoints of the subintervals are introduced as additional parameters, and problem (1.1), (1.2) is reduced to an equivalent problem with parameters. Employing a regular partition reduced the problem to a system of linear algebraic equations in parameters. It was proved that the solvability of this system is equivalent to that of problem (1.1), (1.2). This fact enabled that to establish criteria for the unique solvability of linear boundary value problems for Fredholm integro-differential equations. The construction of a system of linear algebraic equations is the basis for methods and algorithms for solving problem (1.1), (1.2). The Dzhumabaev's parametrization method is developed, in particular, in the works [21]–[27].

Let $X(t)$ is the fundamental matrix of the differential equation $\dfrac{dX(t)}{dt} = A(t)X(t)$. Then from the equation (1.1) we obtain

$$x(t) = X(t) \int_0^t X^{-1}(s)\Big(A(s)x(s) + B(s) \max\big\{x(\tau) : \tau \in \big[\delta_1(s); \delta_2(s)\big]\big\}\Big) ds +$$

$$+ X(t) \int_0^t X^{-1}(s) \int_0^T K(s,\theta)x(\theta) d\theta ds + X(t) \int_0^t X^{-1}(s) f(s) ds. \tag{1.3}$$

The unique solvability of the equation (1.3) we prove in the space $BD\big([0,T], \mathbb{R}^n\big)$.

Solution to problem (1.1), (1.2) is continuously differentiable on $(0,T)$ and satisfying the differential equation (1.1) and boundary condition (1.2).

2 Existence and uniqueness of solution of the equation (1.3)

We use the Banach space $BD\big([0,T], \mathbb{R}^n\big)$ with the norm

$$\| x(t) \|_{BD[0,T]} = \| x(t) \|_1 + h \left\| x'(t) \right\|_1,$$

where $0 < h = \max\limits_{0 \le t \le T} \big| \delta_1(t) \big| + \max\limits_{0 \le t \le T} \big| \delta_2(t) \big|$.

In our further calculations need in the following lemma.

Lemma 2.1 ([28]). *For the difference of two functions with maxima there holds the following estimate*

$$\big\| \max\big\{x(\tau) : \tau \in [t-h,t]\big\} - \max\big\{y(\tau) : \tau \in [t-h,t]\big\}\big\|_1 \leq$$

$$\leq \|\, x(t) - y(t) \,\|_1 + h \Big\| \frac{\partial}{\partial t}\big[x(t) - y(t)\big] \Big\|_1,$$

where $0 < h = \text{const}$.

For the equation (1.3) we consider the following iteration process:

$$x^0(t) = g(t) \equiv X(t) \int_0^t X^{-1}(s) f(s) ds, \quad t \in [0,T],$$

$$x^{k+1}(t) = g(t) + X(t) \int_0^t X^{-1}(s) \Big(A(s) x^k(s) + B(s) \max\big\{x^k(\tau) : \tau \in \big[\delta_1(s), \delta_2(s)\big]\big\}\Big) ds +$$

$$+ X(t) \int_0^t X^{-1}(s) \int_0^T K(s,\theta) x^k(\theta) d\theta ds, \tag{2.1}$$

where $k = 0, 1, 2, ...$

Theorem 2.1. *If the following conditions be fulfilled*

$$\|\, X(t) \,\|_1 \int_0^t \|\, X^{-1}(s) \,\|_1 \max \Big\{ \|\, A(s) \,\|_1 ; \|\, B(s) \,\|_1 ; \int_0^T \|\, K(s,\theta) \,\|_1 d\theta \Big\} ds = C_1 < \infty,$$

where $0 < C_1 = \text{const} < \infty$, *and*
$\rho = \max\big\{C_2; h\, C_3\big\} < 1$, *then the functional-integral equation* (1.3) *has a unique solution in the class* $BD\big([0,T], \mathbb{R}^n\big)$, $0 < C_2 = \text{const} < \infty$ *and* $0 < C_3 = \text{const} < \infty$ *define from the formulas* (2.5) *and* (2.6) *below.*

Proof. We use the iteration process (2.1). Then we obtain that the following estimates are true:

$$\big\|x^0(t)\big\|_1 \leq \|g(t)\|_1 = \|\, X(t) \,\|_1 \int_0^t \|\, X^{-1}(s) \,\|_1 \|f(s)\|_1 ds = g_0 = \text{const} < \infty, \tag{2.2}$$

$$\big\|x^{k+1}(t) - x^k(t)\big\|_1 \leq \|\, X(t) \,\|_1 \int_0^t \|\, X^{-1}(s) \,\|_1 \Big[\|A(s)\|_1 \times$$

$$\times \big\|x^k(s) - x^{k-1}(s)\big\|_1 + \|B(s)\|_1 \big\| \max\big\{x^k(\tau) : \tau \in [s-h, s]\big\} -$$

$$- \max\big\{x^{k-1}(\tau) : \tau \in [s-h, s]\big\}\big\|_1 + \int_0^T \|\, K(s,\theta) \,\|_1 \big\|x^k(\theta) - x^{k-1}(\theta)\big\|_1 d\theta \Big] ds \leq$$

$$\leq C_1 \Big[2 \big\|x^k(t) - x^{k-1}(t)\big\|_1 +$$

$$+\big\| \max\big\{x^k(\tau):\tau\in[t-h,t]\big\}-\max\big\{x^{k-1}(\tau):\tau\in[t-h,t]\big\}\big\|_1\Big].$$

Applying the Lemma 2.1 to the last inequality, we obtain

$$\big\|x^{k+1}(t)-x^k(t)\big\|_1\le C_1\Big[3\big\|x^k(t)-x^{k-1}(t)\big\|_1+h\Big\|\frac{d}{dt}\Big(x^k(t)-x^{k-1}(t)\Big)\Big\|_1\Big]. \tag{2.3}$$

Similarly, from the equation (1.1) we derive

$$\Big\|\frac{d}{dt}\big(x^{k+1}(t)-x^k(t)\big)\Big\|_1\le \|A(t)\|_1\|x^k(t)-x^{k-1}(t)\|_1+\int\limits_0^T\|K(t,s)\|_1\big\|x^k(s)-x^{k-1}(s)\big\|_1 ds+$$

$$+\|B(t)\|_1\big\|\max\big\{x^k(\tau):\tau\in[t-h,t]\big\}-\max\big\{x^{k-1}(\tau):\tau\in[t-h,t]\big\}\big\|_1\le$$

$$\le\Big(\|A(t)\|_1+\|B(t)\|_1+\int\limits_0^T\|K(t,s)\|_1 ds\Big)\big\|x^k(t)-x^{k-1}(t)\big\|_1+$$

$$+h\|B(t)\|_1\Big\|\frac{d}{dt}\big(x^k(t)-x^{k-1}(t)\big)\Big\|_1. \tag{2.4}$$

We denote that

$$C_2=\max\Big\{2C_1;\|A(t)\|_1+\|B(t)\|_1+\int\limits_0^T\|K(t,s)\|_1 ds\Big\}, \tag{2.5}$$

$$C_3=\max\Big\{C_1;\|B(t)\|_1\Big\}. \tag{2.6}$$

Then from estimates (2.3) and (2.4) we get

$$\big\|x^{k+1}(t)-x^k(t)\big\|_{BD[0,T]}\le\rho\big\|x^k(t)-x^{k-1}(t)\big\|_{BD[0,T]}, \tag{2.7}$$

where $\rho=\max\big\{C_2;h\,C_3\big\}<1$.

It follows from the estimates (2.2) and (2.7) that the operator on the right-hand side of (1.3) is contraction mapping and the equation (1.3) has a unique solution in the space $BD\big([0,T],\mathbb{R}^n\big)$. The theorem is proved. □

3 Schemes of calculation of the solution

Take some step $h_0>0$, where $Nh_0=T$ ($N\in\mathbb{N}$) and make a partition of interval $[0,T)$:

$$[0,T)=\bigcup_{r=1}^{N}\big[(r-1)h_0,rh_0\big).$$

We will denote by $C\big([0,T],h_0,\mathbb{R}^{nN}\big)$ the Banach space of continuous vector-functions $x(t)=\big(x_1(t),\,x_2(t),\dots,x_N(t)\big)$ with the norm

$$\|x(t)\|_2=\max_{r=1:N}\sup_{t\in\big[(r-1)h_0,rh_0\big)}|x_r(t)|,$$

where $x_r : \big[(r-1)h_0, rh_0\big) \to \mathbb{R}^n$, and $\lim\limits_{t\to rh_0-0} x_r(t)$ for all $r=\overline{1,N}$ is finite.

We put the parameters $\lambda_r = x_r\big((r-1)h_0\big)$ and use the substitutions $u_r(t) = x_r(t) - \lambda_r$ on each $r-$th subinterval. Then we have the following boundary value problem:

$$\frac{du_r(t)}{dt} = A(t)\big(u_r(t)+\lambda_r\big) + B(t)\max\Big\{\big(u_r(\tau)+\lambda_r\big) : \tau\in\big[\delta_1(t);\delta_2(t)\big]\Big\}+$$

$$+\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j} K(t,s)\big(u_j(s)+\lambda_j\big)ds + f(t),\;\; t\in\big[t_{r-1},t_r\big),\;\; r=\overline{1,N}, \tag{3.1}$$

$$u_r\big((r-1)h_0\big)=0,\quad r=\overline{1,N}, \tag{3.2}$$

$$B_0\big(u_1(0)+\lambda_1\big) + C_0\lim_{t\to T-0}u_N(t) + C_0\lambda_N = D_0,\;\; D_0\in\mathbb{R}^n, \tag{3.3}$$

$$\lim_{t\to t_p-0}u_p(t)+\lambda_p = u_{p+1}\big(t_p\big)+\lambda_{p+1},\;\; p=\overline{1,N-1}. \tag{3.4}$$

Introduction of additional parameters provides us with the initial data (3.2). For fixed values of $\lambda\in\mathbb{R}^{nN}$, solving the special Cauchy problem (3.1), (3.2) for integro-differential equations, we obtain the function $u(t)$. The problem (3.1), (3.2) we write as the equivalent system of integral equations

$$u_r(t) = X(t)\int\limits_{(r-1)h_0}^{t} X^{-1}(\theta_1)\big[A(\theta_1)+B(\theta_1)\big]d\theta_1\lambda_r + X(t)\int\limits_{(r-1)h_0}^{t} X^{-1}(\theta_1)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}K(\theta_1,s)dsd\theta_1\lambda_j+$$

$$+X(t)\int\limits_{(r-1)h_0}^{t} X^{-1}(\theta_1)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}K(\theta_1,s)dsd\theta_1\lambda_j + X(t)\int\limits_{\xi_r}^{t}X^{-1}(\theta_1)f(\theta_1)d\theta_1+$$

$$+X(t)\int\limits_{(r-1)h_0}^{t} X^{-1}(\theta_1)A(\theta_1)u_r(\theta_1)d\theta_1 + X(t)\int\limits_{(r-1)h_0}^{t} X^{-1}(\theta_1)B(\theta_1)\max\Big\{u_r(\tau) : \tau\in\big[\delta_1(\theta_1);\delta_2(\theta_1)\big]\Big\}d\theta_1+$$

$$+X(t)\int\limits_{(r-1)h_0}^{t} X^{-1}(\theta_1)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}K(\theta_1,s)u_j(\theta_1)dsd\theta_1,\;\; t\in\big[(r-1)h_0,rh_0\big),\;\; r=\overline{1,N}. \tag{3.5}$$

For further continuation, we suppose that $P(t)$ be an arbitrary square matrix continuous on the interval $\big[(r-1)h_0, rh_0\big)$ and it has a finite limit $\lim\limits_{t\to rh_0-0}P(t)$, $r=\overline{1,N}$. Take a numeric $\nu\in\mathbb{N}$ and denote by $\Phi_{\nu,r}\big(A(\cdot),P(\cdot),t\big)$ the sum

$$\int\limits_{(r-1)h_0}^{t}P(s_1)ds_1 + \int\limits_{(r-1)h_0}^{t}A(s_1)\int\limits_{(r-1)h_0}^{s_2}P(s_2)ds_2ds_1+\ldots+$$

$$+\int\limits_{(r-1)h_0}^{t}A(s_1)\ldots\int\limits_{(r-1)h_0}^{s_{\nu-2}}A(s_{\nu-1})\int\limits_{(r-1)h}^{s_{\nu-1}}P(s_\nu)ds_\nu ds_{\nu-1}\ldots ds_1,\;\; t\in\big[(r-1)h_0,rh_0\big),\;\; r=\overline{1,N}.$$

The sum $\Phi_{\nu,r}(A(\cdot),P(\cdot),t)$ is continuous on $\big[(r-1)h_0, rh_0\big)$ and has a finite limit

$$\lim_{t\to rh_0-0}\Phi_{\nu,r}\big(A(\cdot),P(\cdot),t\big)=\Phi_{\nu,r}\big(A(\cdot),P(\cdot),rh_0\big) \text{ for all } nu\in\mathbb{N},\ \ r=\overline{1,N}.$$

It is evident that $\Phi_{*,r}\big(A(\cdot),P(\cdot),t\big)=\lim\limits_{\nu\to\infty}\Phi_{\nu,r}(A(\cdot),P(\cdot),t)$ is a sum of uniformly convergent series on $\big[(r-1)h_0, rh_0\big)$, and the sum is continuous on the interval $\big[(r-1)h_0, rh_0\big)$ and has a finite limit

$$\lim_{t\to rh_0-0}\Phi_{*,r}\big(A(\cdot),P(\cdot),t\big)=\Phi_{*,r}\big(A(\cdot),P(\cdot),rh_0\big),\quad r=\overline{1,N}.$$

Substituting the right-hand side of (3.5) into $u_r(s)$ in (3.5) and repeating the process ν $(\nu\in\mathbb{N})$ times, we obtain the following presentation for a function $u_r(t)$:

$$u_r(t)=F_{\nu,r}(t)\lambda_r+G_{\nu,r}(u_r,t)+H_{\nu,r}(u_r,t)+$$

$$+\Psi_{\nu,r}(u_r,t)+K_{\nu,r}(t),\ \ t\in\big[(r-1)h_0,rh_0\big),\ \ r=\overline{1,N}, \tag{3.6}$$

where

$$F_{\nu,r}(t)=\Phi_{\nu,r}\big(A(\cdot),A(\cdot)+B(\cdot),t\big), K_{\nu,r}(t)=\Phi_{\nu,r}\big(A(\cdot),f(\cdot),t\big), H_{\nu,r}(t)=\Phi_{\nu,r}\big(A(\cdot),B(\cdot)\max\{u_r(\tau)\},t\big),$$

and

$$\Psi_{\nu,r}(u_r,t)=\int\limits_{(r-1)h_0}^{t}A(s_1)\ldots\int\limits_{(r-1)h_0}^{s_{\nu-1}}\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}K\big(\theta,s_\nu\big)u_j(\theta)\theta ds_\nu\ldots ds_1,$$

$$H_{\nu,r}(u_r,t)=\int\limits_{(r-1)h_0}^{t}A(s_1)\ldots\int\limits_{(r-1)h_0}^{s_{\nu-1}}B(s_\nu)\max\Big\{u_r(\tau_\nu):\tau\in\big[s_\nu-h,s_\nu\big]\Big\}ds_\nu\ldots ds_1,$$

$$G_{\nu,r}(u_r,t)=\int\limits_{(r-1)h_0}^{t}A(s_1)\ldots\int\limits_{(r-1)h_0}^{s_{\nu-1}}A(s_\nu)u_r(s_\nu)ds_\nu\ldots ds_1,\quad t\in\big[(r-1)h_0,rh_0\big),\ \ r=\overline{1,N}.$$

Determine $\lim\limits_{t\to rh_0-0}u_r(t),\ r=\overline{1,N}$ from the formula (3.6). Substituting the appropriate expressions into (3.3) and (3.4), pre-multiplying (3.3) by $h_0>0:\ Nh_0=T$, we obtain a system of linear algebraic equations with respect to parameters:

$$Q_\nu(h_0)\lambda=-K_\nu(h_0)-G_\nu(u,h_0)-H_\nu(u,h_0)-\Psi_{\nu,r}(u,h_0),\quad \lambda\in\mathbb{R}^{nN},$$

where $Q_\nu(h_0)=$

$$\begin{pmatrix} h_0B_0 & O & O & \ldots & O & h_0C_0\big(I+F_{\nu,N}(Nh_0)\big)\\ I+K_{\nu,1}(h_0) & -I & O & \ldots & O & O\\ O & I+K_{\nu,2}(2h_0) & -I & \ldots & O & O\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ O & O & O & \ldots & -I & O\\ O & O & O & \ldots & I+K_{\nu,N-1}\big((N-1)h_0\big) & -I \end{pmatrix},$$

$I:\mathbb{R}^n\to\mathbb{R}^n$ is the identity matrix, $O:\mathbb{R}^n\to\mathbb{R}^n$ is a zero matrix,
$K_\nu(h_0)=\big(-h_0D_0+h_0C_0F_{\nu,N}(Nh_0),\ K_{\nu,1}(h_0),\ldots,K_{\nu,N-1}((N-1)h_0)\big)\in\mathbb{R}^{nN},$

$G_\nu(u,h_0) = \big(h_0 C_0 G_{\nu,N}(u_N, Nh_0), G_{\nu,1}(u_1,h_0), \ldots, G_{\nu,N-1}\big(u_{N-1},(N-1)\big)\big) \in \mathbb{R}^{nN}$.
Similarly it is defined $H_\nu(u,h_0)$ and $\Psi_{\nu,r}(u,h_0)$.

It is clear that we find the solution $\big(\lambda, u(t)\big)$ to the multi-point boundary value problem with parameters (3.1)–(3.4), where $\lambda \in \mathbb{R}^{nN}$, $u(t) = \big(u_1(t), u_2(t), \ldots, u_N(t)\big)$. Suppose that for given ν, h_0 the matrix $Q_\nu(h_0) : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ has an inverse one.
a) Find the initial approximation on parameter $\lambda^{(0)} = \big(\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots,\ \lambda_N^{(0)}\big) \in \mathbb{R}^{nN}$ solving the system of functions $Q_\nu(h_0)\lambda = -F_\nu(h)$.
b) Determine the components of function system $u^{(0)}(t) = \big(u_1^{(0)}(t), u_2^{(0)}(t), \ldots,\ u_N^{(0)}(t)\big)$ according to the formulas

$$u_r^{(0)}(t) = F_{\nu,r}(t)\lambda_r^{(0)} + K_{\nu,r}(t), \quad t \in \big[(r-1)h_0, rh_0\big), \quad r = \overline{1,N}.$$

c) Find the next approximation on parameter $\lambda^{(1)} = \big(\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots,\ \lambda_N^{(1)}\big) \in \mathbb{R}^{nN}$ solving the system of equations $Q_\nu(h_0)\lambda = -K_\nu(h_0) - G_\nu(u^{(0)}, h_0) - H_\nu\big(u^{(0)}, h_0\big)$.
d) Determine the components of function system $u^{(1)}(t) = \big(u_1^{(1)}(t), u_2^{(1)}(t), \ldots,\ u_N^{(1)}(t)\big)$ according to the formulas

$$u_r^{(1)}(t) = F_{\nu,r}(t)\lambda_r^{(1)} + K_{\nu,r}(t) + G_{\nu,r}\big(u_r^{(0)}, t\big) + H_{\nu,r}\big(u_r^{(0)}\big), \quad t \in \big[(r-1)h_0, rh_0\big), \quad r = \overline{1,N}.$$

And so on. Continuing this process, at the $k-$th step of algorithm we obtain the pair $\big(\lambda^{(k)}, u^{(k)}(t)\big)$, $k = 0, 1, \ldots$.

4 Main result

In light of equivalence of problems (1.1), (1.2) and (3.1)–(3.4), the Theorem 2.1 leads to

Theorem 4.1. *The boundary value problem* (1.1), (1.2) *is uniquely solvable, if only if for given* $h > 0 : Nh = T \ (N \in \mathbb{N}),\ \chi \in (0,1]$ *there exists* $\nu = \nu(h)$ $(\nu \in \mathbb{N})$ *such that the matrix* $Q_\nu(h) : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ *is invertible, and the conditions of the theorem 2.1 are fulfilled.*

Funding The authors have no funding.
Data availability This manuscript has no associated data.
Ethical Conduct Not applicable.
Conflicts of interest The authors declare that there is no conflict of interest.

References

1. A. Bressan, W. Shen, "A semigroup approach to an integro-differential equation modeling slow erosion", *Journal of Differential Equations* **257** (7), 2360–2403 (2014). https://doi.org/10.1016/j.jde.2014.05.038.

2. M. Dehghan, "Solution of a partial integro-differential equation arising from viscoelasticity", *International Journal of Computational Mathematics* **83** (1), 123–129 (2006). https://doi.org/10.1080/00207160500069847.

3. M. Dehghan, F. Shakeri, "Solution of parabolic integro-differential equations arising in heat conduction in materials with memory via He's variational iteration technique", *International Journal for Numerical Methods in Biomedical Engineering* **26** (6), 705–715 (2010). https://doi.org/10.1002/cnm.1166.

4. V. Volpert, *Elliptic partial differential equations*. Springer, Basel, 2014. https://doi.org/10.1007/978-3-0348-0813-2.

5. M. I. Berenguer, D. Gamez, A. J. Lopez Linares, "Fixed point techniques and Schauder bases to approximate the solution of the first order nonlinear mixed Fredholm–Volterra integro-differential equation", *Journal of Computation for Applied Mathematics*, **252**, 52–61 (2013). https://doi.org/10.1016/j.cam.2012.09.020

6. A. A. Boichuk, A. M. Samoilenko, *Generalized inverse operators and Fredholm boundary-value problems*. VSP, Utrecht, Boston, 2004.

7. S. Kheybari, M. T. Darvishi, A. M. Wazwaz, "A semi-analytical approach to solve integro-differential equations", *Journal of Computation for Applied Mathematics* **317**, 17–30 (2017). https://doi.org/10.1016/j.cam.2016.11.011

8. V. Lakshmikantham, M. R. Rao, *Theory of integro-differential equations*. Gordon Breach, London, 1995.

9. A. M. Wazwaz, *Linear and nonlinear integral equations: Methods and applications*. Higher Education Press, Beijing and Springer-Verlag, Berlin, Heidelberg, 2011.

10. S. Yüzbasi, "A collocation method based on Bernstein polynomials to solve nonlinear Fredholm–Volterra integro-differential equations", *Applied Mathematics Computation* **273**, 142–154 (2016). https://doi.org/10.1016/j.amc.2015.09.091

11. T. K. Yuldashev, "On the solvability of a boundary value problem for the ordinary Fredholm integrodifferential equation with a degenerate kernel", *Computational Mathematics and Mathematical Physics* **59** (2), 241–252 (2019). https://doi.org/10.1134/S0965542519020167

12. T. K. Yuldashev, "On inverse boundary value problem for a Fredholm integro-differential equation with degenerate kernel and spectral parameter", *Lobachevskii Journal of Mathematics* **40** (2), 230–239 (2019). https://doi.org/10.1134/S199508021902015X

13. T. K. Yuldashev, "Determination of the coefficient and boundary regime in boundary value problem for integro-differential equation with degenerate kernel", *Lobachevskii Journal of Mathematics* **38** (3), 547–553 (2017). https://doi.org/10.1134/S199508021703026X

14. T. K. Yuldashev, "On a boundary-value problem for a fourth-order partial integro-differential equation with degenerate kernel", *Journal of Mathematical Sciences* **245** (4), 508–523 (2020). https://doi.org/10.1007/s10958-020-04707-2

15. T. K. Yuldashev, T. G. Ergashev, T. A. Abduvahobov, "Nonlinear system of impulsive integro-differential equations with Hilfer fractional operator and mixed maxima", *Chelyabinsk Phisical and Mathematical Journal* **7** (3), 312–325 (2022).

16. D. S. Dzhumabaev, "A method for solving the linear boundary value problem for an integro-differential equation", *Computational Mathematics and Mathematical Physics* **50** (7), 1150–1161 (2010). https://doi.org/10.1134/S0965542510070043

17. D. S. Dzhumabaev, "Necessary and sufficient conditions for the solvability of linear boundary-value problems for the Fredholm integro-differential equations", *Ukrainian Mathematical Journal* **66** (8), 1200–1219 (2015). https://doi.org/10.1007/s11253-015-1003-6

18. D. S. Dzhumabaev, "On one approach to solve the linear boundary value problems for Fredholm integro-differential equations", *Journal of Computational and Applied Mathematics* **294**, 342–357 (2016). https://doi.org/10.1016/j.cam.2015.08.023

19. D. S. Dzhumabaev, "New general solutions to linear Fredholm integro-differential equations and their applications on solving the boundary value problems", *Journal of Computational and Applied Mathematics* **327**, 79–108 (2018). https://doi.org/10.1016/j.cam.2017.06.010

20. D. S. Dzhumabaev, "Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation", *Journal of Computational and Applied Mathematics* **29**, 34–46 (1989). https://doi.org/10.1016/0041-5553(89)90038-4

21. A. T. Asanova, "On a nonlocal boundary-value problem for systems of impulsive hyperbolic equations", *Ukrainian Mathematical Journal* **65** (3), 349–365 (2013). https://doi.org/10.1007/s11253-013-0782-x.

22. A. T. Assanova, "Solvability of a nonlocal problem for a hyperbolic equation with integral conditions", *Electronic Journal of Differential Equations* **2017** (170), 1–12 (2017).

23. A. T. Assanova, "An integral-boundary value problem for a partial differential equation of second order", *Turkish Journal of Mathematics* **43** (4), 1967–1978 (2019). https://doi.org/10.3906/mat-1903-111.

24. A. T. Assanova, "On the solvability of nonlocal problem for the system of Sobolev-type differential equations with integral condition", *Georgian Mathematical Journal* **28** (1), 49–57 (2021). https://doi.org/10.1515/gmj-2019-2011.

25. A. T. Assanova, E. A. Bakirova, Z. M. Kadirbayeva, R. E. Uteshova, "A computational method for solving a problem with parameter for linear systems of integro-differential equations", *Computational and Applied Mathematics* **39** (3), Art.no. 248 (2020). https://doi.org/10.1007/s40314-020-01298-1.

26. A. T. Assanova, "A generalized integral problem for a system of hyperbolic equations and its applications", *Hacettepe Journal of Mathematics and Statistics* **52** (6), 1513–1532 (2023). https://doi.org/10.15672/hujms.1094454.

27. A. T. Assanova, S. T. Mynbayeva, "New general solution to a quasilinear Fredholm integro-differential equation and its application", *Lobachevskii Journal of Mathematics* **44** (10), 4231–4239 (2023). https://doi.org/10.1134/S1995080223100062.

28. T. K. Yuldashev, A. K. Fayziyev, "Periodic solutions of impulsive system of equations with a nonlinear function under the sign of a second-order differential and maxima", *Lobachevskii Journal of Mathematics* **46** (2), 658–671 (2025). https://doi.org/10.1134/S1995080225600256