Journal of Osh University "Differential Equations", 2026, Volume 1, Issue 1, P. 1–14.

UDC 517.956.46

Inverse Problem for a Nonlinear Pseudiparabolic Differential Equation with Final Condition and Gerasimov–Caputo Operator

Minzilya T. Kosmakova

Karaganda Buketov University, Universitet Street, 28, 100028 Karaganda, Kazakhstan

E-mail: svetlanamir578@gmail.com, https://orcid.org/0000-0003-4070-0215

Tursun K. Yuldashev*

Tashkent State Transport University, 1, Temiryo'lchilar Street, Tashkent, Ozbekistan

V. I. Romanovskii Institute of Mathematics of Ozbekistan Academy of Science, Tashkent, Ozbekistan

Osh State University, 331, Lenin street, Osh, Kyrgyzstan

E-mail: tursun.k.yuldashev@gmail.com, https://orcid.org/0000-0002-9346-5362

Marjona U. Sagdullaeva

Samarkand State University, University blv, 15, Samarkand, 140104 Ozbekistan

E-mail: marjonasagdullayeva2000@gmail.com, https://orcid.org/0009-0003-8138-2062

Received: November 10, 2025; **First on-line published**: January 10, 2026

Abstract. *In this paper in rectangle domain an inverse problem for a fractional analogue of the pseudoparabolic differential operator with mixed conditions, degeneration and identification source is considered. Fractional operator is the Gerasimov–Caputo type and the solution of the nonlinear differential equation with two spatial variables is studied in the class of generalized functions. The nonlinear Fourier series method is used and by the aid of Kilbas–Saigo function a nonlinear countable system of functional integral equation is obtained. In the proof of unique solvability of the countable system is applied the method of successive approximations in combination with the method of compressing mapping. We use the Cauchy–Schwarz inequality and the Bessel inequality in proving the absolute and uniform convergence of the obtained Fourier series. Then we derive the desire redefinition function also in the form of Fourier series.*

Key words: *Inverse problem, fractional analogue, nonlinear pseudoparabolic differential equation, final condition, degeneration, identification source.*

MSC 2020: 35A01, 35A02, 35N25.

Introduction. Problem statement

Some problems of mechanics turn out to be initial-boundary (mixed) problems. Many mixed problems are studied in solving different problems of hydrodynamics [1]. In [2], [3] mixed problems for linear differential equations of parabolic and hyperbolic types were studied. In works [4, 5], mixed problems for nonlinear differential and integro-differential equations of the second, fourth and higher orders were studied. The main equations of the theory of non-stationary filtration in fractured-pore formations are formulated in the work of G. I. Barenblatt, Yu. P. Zheltov and I. N. Kochina [6] (see also [7]) and, further, developed by many authors [8]–[13]. The theory and applications of fractional calculus have been developed by many authors (see, for example, [14]–[22]). Investigation of the well-known fractional order differentiation operators of Riemann–Liouville type and Gerasimov–Caputo type are important, because they describe diffusion processes [15, vol. 1, 47–85]. A physical and engineering interpretations of the generalized fractional operators are given in [15, vol. 4–8]. Note that boundary value conditions of the type (0.3) (see, below) take place in modeling problems of the flow around a profile by a subsonic velocity stream with a supersonic zone. Different mixed and boundary value problems for differential and integro-differential equations with identification source were studied in the works of many authors (see, for example, in the works [23]–[43]).

In the domain $\Omega = \left\{(t,x,y)\,|\,0<t<T,\ 0<x,y<l\right\}$ a partial differential equation of the following form

$$\left[{}_C D_{0t}^{\alpha} - {}_C D_{0t}^{\alpha}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - t^{\beta}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right] U(t,x,y) = f(t,x,y,U(t,x,y)) \tag{0.1}$$

is considered with final point integral condition

$$U(T,x,y) = \varphi(x,y) + \int\limits_0^T R(s)U(s,x,y)ds, \quad 0\le x,y\le l, \tag{0.2}$$

where β, T and l are given positive real numbers, for $0<\alpha\le 1$ [15, Vol. 1, p. 34]

$${}_C D_{0t}^{\alpha}\eta(t) = I_{0t}^{1-\alpha}\eta'(t) = \frac{1}{\Gamma(1-\alpha)}\int\limits_0^t \frac{\eta'(s)}{(t-s)^{\alpha}}ds, \quad {}_C D_{0t}^{1}\eta(t) = \eta'(t), \quad t\in(0,T)$$

is Gerasimov–Caputo type fractional operator [44, p. 112]

$$I_{0t}^{\alpha}\eta(t) = \frac{1}{\Gamma(\alpha)}\int\limits_0^t \frac{\eta(s)ds}{(t-s)^{1-\alpha}}, \quad t\in(0,T)$$

is Riemann–Liouville integral operator, $a(t), b(t)\in C[0,T]$ is given function, $f(x,y)$ is given function, $\varphi(x,y)\in L_2\big([0,l]^2\big)$ is redefinition function. Sometimes, we use the notations: $\Omega_T\equiv[0,T]$, $\Omega_l\equiv[0,l]$.

We denote by $W_{2,t,x}^{n,4nm}(\overline{\Omega})$ the class of continuously functions $U(t,x)$ of two variables in closed rectangle $\overline{\Omega} = \left\{(t,x)\,|\,0\le t\le T,\ 0\le x\le l\right\}$ and having partial derivatives in it

$$\frac{\partial U(t,x)}{\partial x},\ \frac{\partial^2 U(t,x)}{\partial x^2},\ \ldots,\ \frac{\partial^{4nm} U(t,x)}{\partial x^{4nm}},\ \frac{\partial U(t,x)}{\partial t},\ \ldots,\ \frac{\partial^{n} U(t,x)}{\partial t^{n}},$$

each of them belongs $L_2(\overline{\Omega})$, and n,m are positive integers.

Problem 0.1. Find the generalized solution $U(t,x,y)\in W_{2,t,x,y}^{2,\alpha,2,2}(\overline{\Omega})$, which satisfies partial differential equation (0.1), final value integral condition (0.2), boundary value conditions

$$U(t,0,y) = U(t,l,y) = U(t,x,0) = U(t,x,l) = 0 \tag{0.3}$$

almost everywhere.

Problem 0.2. Find the pair of functions $U(t,x,y)\in W^{2,\alpha,2,2}_{2,t,x,y}(\overline{\Omega})$ $\varphi(x,y)\in L_2\big([0,l]^2\big)$, first of which satisfies the additional condition

$$\int\limits_0^T H(s)U(s,x,y)ds=\psi(x,y), \tag{0.4}$$

where $\psi(x,y)\in L_2\big([0,l]^2\big)$ is given function and $\psi(0,y)=\psi(l,y)=\psi(x,0)=\psi(x,l)=0$.

We assume that for the given functions are true the following boundary conditions

$$\varphi(0,y)=\varphi(l,y)=\varphi(x,0)=\varphi(x,l)=0,$$

$$f(t,0,y,U(t,0,y))=f(t,l,y,U(t,l,y))=f(t,x,0,U(t,x,0))=f(t,x,l,U(t,x,l))=0.$$

1 Reducing the problem 0.1 into countable system of nonlinear integral equations

Nontrivial solutions of the problem (0.1)–(0.3) are sought as a Fourier series

$$U(t,x,y)=\sum_{n,m=1}^{\infty}u_{n,m}(t)\,\vartheta_{n,m}(x,y) \tag{1.1}$$

and we suppose that

$$f(t,x,y,U)=\sum_{n,m=1}^{\infty}f_{n,m}(t,\cdot)\,\vartheta_{n,m}(x,y), \tag{1.2}$$

where

$$u_{n,m}(t)=\int\limits_0^l\int\limits_0^l U(t,\eta,\xi)\,\vartheta_{n,m}(\eta,\xi)d\eta d\xi,\ \ f_{n,m}(t,\cdot)=\int\limits_0^l\int\limits_0^l f(t,\eta,\xi,\cdot)\,\vartheta_{n,m}(\eta,\xi)d\eta d\xi, \tag{1.3}$$

$$\vartheta_{n,m}(x,y)=\frac{2}{l}\sin\lambda_n x\,\sin\lambda_m y=\frac{2}{l}\sin\frac{\pi\,n}{l}x\,\sin\frac{\pi\,m}{l}y,\quad n,m=1,\,2,\,\ldots$$

Substituting Fourier series (1.1) and (1.2) formally into partial differential equation (0.1), we obtain a countable system of ordinary fractional differential equations of $0<\alpha<1$-order with degeneration

$${}_CD^{\alpha}_{0t}u_{n,m}(t)+\mu^2_{n,n}t^{\beta}u_{n,m}(t)=\frac{1}{1+\lambda^2_{n,m}}f_{n,m}\left(t,\,\sum_{i,j=1}^{\infty}u_{i,j}(t)\,\vartheta_{i,j}(x,y)\right), \tag{1.4}$$

where

$$\mu^2_{n,m}=\frac{\lambda^2_{n,m}}{1+\lambda^2_{n,m}},\quad \lambda^2_{n,m}=\left(\frac{\pi\,n}{l}\right)^2(n^2+m^2).$$

We suppose also that there is true the following Fourier expansion

$$\varphi(x,y)=\sum_{n,m=1}^{\infty}\varphi_{n,m}\vartheta_{n,m}(x,y),$$

where

$$\varphi_{n,m}=\int\limits_0^l\int\limits_0^l \varphi(\eta,\xi)\,\vartheta_{n,m}(\eta,\xi)d\eta d\xi. \tag{1.5}$$

By Fourier coefficients (1.3) and (1.5), the final value integral condition (0.2) takes the form

$$u_{n,m}(T)=\varphi_{n,m}+\int\limits_0^T R(s)u_{n,m}(s)ds. \tag{1.6}$$

We use the well known Kilbas–Saigo type function, which is generalized two-parametric Mittag–Leffler function [15, Vol. 1. pp. 269–295]

$$E_{\alpha,\beta}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha\, m+\beta)},\quad z,\alpha,\beta\in\mathbb{C},\quad \mathrm{Re}\,(\alpha)>0.$$

The Kilbas–Saigo function is defined for real $\alpha,m,l\in\mathbb{R}$ and complex $l\in\mathbb{C}$ is defined by the following form [14, 16]

$$E_{\alpha,m,l}(z)=\sum_{k=0}^{\infty}c_k z^k,\quad c_0<1,\quad c_k=\prod_{j=1}^{k-1}\frac{\Gamma(\alpha[jm+l]+1)}{\Gamma(\alpha[jm+l+1]+1)},\quad k=1,2,\,...$$

The Kilbas–Saigo functions belong to the class of entire functions in the complex plane [45].

Let us consider the final value problem for a countable system of ordinary differential equation of fractional order with degeneration

$$_C D_{0t}^{\alpha}u_{n,m}(t)=-\mu_{n,m}^2 t^{\beta}u_{n,m}(t)+f_{n,m}(t,\cdot),\quad u_{n,m}(0)=h_{n,m}, \tag{1.7}$$

where $\beta\in\mathbb{R},\quad 0<\mu_{n,m}^2<1,$

$$h_{n,m}=\int\limits_0^l\int\limits_0^l h(\eta,\xi)\,\vartheta_{n,m}(\eta,\xi)d\eta d\xi,\quad h(x,y)\in L_2\big([0,l]^2\big),$$

$$f_{n,m}(t,\cdot)=\int\limits_0^l\int\limits_0^l f\left(t,\eta,\xi,\sum_{i,j=1}^{\infty}u_{i,j}(t)\,\vartheta_{i,j}(\eta,\xi)\right)\vartheta_{n,m}(\eta,\xi)d\eta d\xi.$$

Let $\gamma\in[0,1)$. Then we consider the class of following functions

$$C_{\gamma}[0,T]=\big\{g_{n,m}(t):t^{\gamma}g_{n,m}(t)\in C[0,T]\big\},$$
$$C_{\gamma}^{\alpha}[0,T]=\big\{g_{n,m}(t)\in C[0,T]:\ {}_C D_{0t}^{\alpha}g_{n,m}(t)\in C_{\gamma}[0,T]\big\}.$$

We use the known fact that for $|\arg z|\le\sigma$ and $|z|\ge 0$ the following estimate is true [45]

$$|E_{\alpha,\mu}(z)|\le M_1(1+|z|)^{\frac{1-\delta}{\alpha}}e^{\mathrm{Re}z^{\frac{1}{\alpha}}}+\frac{M_2}{1+|z|}, \tag{1.8}$$

where M_1 and M_2 are constants, not depending from z; $\alpha < 2$, $z \in \mathbb{C}$, δ is real constant and σ is fixed number from the interval $\left(\frac{\pi\alpha}{2}, \min\left\{\pi, \pi\alpha\right\}\right)$. If we put $z = \mu_{n,m}^2 t^\beta (t-\tau)^\alpha$, $\delta = \alpha$, then from (1.8) we have

$$\left| E_{\alpha,\alpha}\left(\mu_{n,m}^2 t^\beta (t-\tau)^\alpha\right) \right| \leq M_1 \left[1 + \mu_{n,m}^2 t^\beta (t-\tau)^\alpha\right]^{\frac{1-\alpha}{\alpha}} e^{\left(\mu_{n,m}^2 t^\beta (t-\tau)^\alpha\right)^{\frac{1}{\alpha}}} +$$

$$+ \frac{M_2}{1 + \mu_{n,m}^2 t^\beta (t-\tau)^\alpha} \leq M_3. \tag{1.9}$$

Lemma 1.1. *Let $\gamma \in [0, \alpha]$, $\beta \geq 0$. Then for all $g_{n,m}(t) \in C_\gamma[0,T]$ there exists a unique solution $u_{n,m}(t) \in C_\gamma^\alpha[0,T]$ of the Cauchy problem (1.7). This solution has the following form*

$$u_{n,m}(t) = \varphi_{n,m} E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}\left(-\mu_{n,m}^2 t^{\alpha+\beta}\right) + \int\limits_0^t K(t,\tau) g_{n,m}(\tau) d\tau, \tag{1.10}$$

where

$$K(t,\tau) = \sum_{i=1}^n K_i(t,\tau), \tag{1.11}$$

$$K_0(t,\tau) = \frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1},\ K_i(t,\tau) = \frac{\mu_{n,m}^2}{\Gamma(\alpha)} \int\limits_\tau^t s^\beta (t-s)^{\alpha-1} K_{i-1}(s,\tau) ds,\ i = 1, 2, \ldots, \tag{1.12}$$

$E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}\left(\mu_{n,m}^2 t^{\alpha+\beta}\right)$ *is Kilbas–Saigo function.*

Moreover, for the kernel (1.11) in the case of $\gamma \in [0, \alpha]$, $\beta \geq 0$ there holds the following estimate

$$|K(t,\tau)| \leq (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\mu_{n,m}^2 t^\beta (t-\tau)^\alpha\right) \leq (t-\tau)^{\alpha-1} M_3, \tag{1.13}$$

where $M_3 = \mathrm{const}$.

Consequently, the general solution of the countable system of ordinary differential equations (1.7) we write as

$$u_{n,m}(t) = C_{n,m} E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}\left(-\mu_{n,m}^2 t^{\alpha+\beta}\right) + \frac{1}{1+\lambda_{n,m}^2} \int\limits_0^t K(t,s)\, f_{n,m}(s,\cdot)\big] ds, \tag{1.14}$$

where $C_{n,m}$ is arbitrary constant, kernel $K(t,s)$ is defined by the formulas (1.11), (1.12) and for the kernel is true the estimate (1.13).

To find the unknown coefficients $C_{n,m}$ in (1.14), we use condition (1.6). So, substituting the equation (1.14) into (1.6), we obtain

$$u_{n,m}(T) = C_{n,m} E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}\left(-\mu_{n,m}^2 T^{\alpha+\beta}\right) + \frac{1}{1+\lambda_{n,m}^2} \int\limits_0^T K(T,s)\, f_{n,m}(s,\cdot) ds =$$

$$= \varphi_{n,m} + \omega \int\limits_0^T R(s) u_{n,m}(s) ds.$$

Hence, we derive

$$C_{n,m}=\varphi_{n,m}Q_T^{-1}+Q_T^{-1}\int\limits_0^T\left\{R(t)\,u_{n,m}(t)-\frac{1}{1+\lambda_{n,m}^2}K(T,t)\,f_{n,m}(t,\cdot)\right\}dt, \tag{1.15}$$

where

$$Q_T=E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left(-\mu_{n,m}^2T^{\alpha+\beta}\right). \tag{1.16}$$

Further, substituting the coefficient presentation's (1.15) into equation (1.14) and taking into account (1.16), we derive a countable system of nonlinear integral equations (CSNIE)

$$u_{n,m}(t)=\varphi_{n,m}\,Q_t\,Q_T^{-1}+Q_t\,Q_T^{-1}\int\limits_0^T R(s)\,u_{n,m}(s)ds+$$

$$+\frac{1}{1+\lambda_{n,m}^2}\int\limits_0^T\overline{K}(t,s)\,f_{n,m}(s,\cdot)ds, \tag{1.17}$$

where

$$\overline{K}(t,s)=\begin{cases}K(t,s)-Q_t\,Q_T^{-1}K(T,s),\; t\le s\le T,\\ K(t,s),\;\; 0\le s\le t.\end{cases}$$

2 Redefinition function. Realisation of the inverse Problem

Substituting the countable system of integral equations (1.17) into Fourier series (1.1), we obtain

$$U(t,x,y)=\sum_{n,m=1}^{\infty}\vartheta_{n,m}(x,y)\Bigg\{\varphi_{n,m}\,Q_t\,Q_T^{-1}+Q_t\,Q_T^{-1}\int\limits_0^T R(s)\,u_{n,m}(s)ds+$$

$$+\frac{1}{1+\lambda_{n,m}^2}\int\limits_0^T\overline{K}(t,s)\,f_{n,m}(s,\cdot)ds\Bigg\}. \tag{2.1}$$

By virtue of (1.1), (1.2) and (1.5), the additional condition (0.4) we rewrite in the form

$$\psi_{n,m}=\int\limits_0^T H(s)u_{n,m}(s)ds.$$

We apply the last condition to the (1.17)

$$\psi_{n,m}=\int\limits_0^T H(s)\Bigg\{\varphi_{n,m}\,Q_s\,Q_T^{-1}+Q_s\,Q_T^{-1}\int\limits_0^T R(\theta)\,u_{n,m}(\theta)d\theta+$$

$$+\frac{1}{1+\lambda_{n,m}^2}\int\limits_0^T\overline{K}(s,\theta)\,f_{n,m}(\theta,\cdot)d\theta\Bigg\}ds. \tag{2.2}$$

Assume that $q_0 = \int\limits_0^T H(s)\, Q_s\, Q_T^{-1} ds \neq 0$. Then from (2.2) we have

$$\varphi_{n,m} = q_0^{-1}\, \psi_{n,m} - \frac{q_0^{-1}}{1+\lambda_{n,m}^2} \int\limits_0^T H(s) \int\limits_0^T \overline{K}(s,\theta)\, f_{n,m}(\theta,\cdot) d\theta ds -$$

$$-q_0^{-1} \int\limits_0^T H(s)\, Q_s\, Q_T^{-1} \int\limits_0^T R(\theta)\, u_{n,m}(\theta) d\theta ds. \tag{2.3}$$

Hence, we obtain the Fourier series for redefinition function

$$\varphi(x,y) = \sum_{n,m=1}^{\infty} \vartheta_{n,m}(x,y) \Bigg\{ q_0^{-1}\, \psi_{n,m} - \frac{q_0^{-1}}{1+\lambda_{n,m}^2} \int\limits_0^T H(s) \int\limits_0^T \overline{K}(s,\theta)\, f_{n,m}(\theta,\cdot) d\theta ds -$$

$$-q_0^{-1} \int\limits_0^T H(s)\, Q_s\, Q_T^{-1} \int\limits_0^T R(\theta)\, u_{n,m}(\theta) d\theta ds \Bigg\}. \tag{2.4}$$

Redefinition function in (2.4) consists unknown function $u_{n,m}(t)$. So, the function (2.3) we substitute into the equation (1.17) and Fourier series (2.1):

$$u_{n,m}(t) = J(t; u_{n,m}(t)) \equiv q_0^{-1}\, Q_t\, Q_T^{-1}\, \psi_{n,m} +$$

$$+\frac{1}{1+\lambda_{n,m}^2} \Bigg[\int\limits_0^T \overline{K}(t,s) \int\limits_0^l \int\limits_0^l F\Bigg(t,\eta,\xi, \sum_{i,j=1}^{\infty} u_{i,j}(t)\, \vartheta_{i,j}(\eta,\xi)\Bigg) \vartheta_{n,m}(\eta,\xi) d\eta d\xi ds -$$

$$-q_0^{-1}\, Q_t\, Q_T^{-1} \int\limits_0^T H(s) \int\limits_0^T \overline{K}(s,\theta) \int\limits_0^l \int\limits_0^l F\Bigg(t,\eta,\xi, \sum_{i,j=1}^{\infty} u_{i,j}(t)\, \vartheta_{i,j}(\eta,\xi)\Bigg) \vartheta_{n,m}(\eta,\xi) d\eta d\xi d\theta ds \Bigg] +$$

$$+Q_t\, Q_T^{-1} \Bigg[\int\limits_0^T R(s)\, u_{n,m}(s) ds - q_0^{-1} \int\limits_0^T H(s)\, Q_s\, Q_T^{-1} \int\limits_0^T R(\theta)\, u_{n,m}(\theta) d\theta ds \Bigg], \tag{2.5}$$

$$U(t,x,y) = \sum_{n,m=1}^{\infty} \vartheta_{n,m}(x,y) \Bigg\{ q_0^{-1}\, Q_t\, Q_T^{-1}\, \psi_{n,m} +$$

$$+\frac{1}{1+\lambda_{n,m}^2} \Bigg[\int\limits_0^T \overline{K}(t,s) \int\limits_0^l \int\limits_0^l F\Bigg(t,\eta,\xi, \sum_{i,j=1}^{\infty} u_{i,j}(t)\, \vartheta_{i,j}(\eta,\xi)\Bigg) \vartheta_{n,m}(\eta,\xi) d\eta d\xi ds -$$

$$-q_0^{-1}\, Q_t\, Q_T^{-1} \int\limits_0^T H(s) \int\limits_0^T \overline{K}(s,\theta) \int\limits_0^l \int\limits_0^l F\Bigg(\theta,\eta,\xi, \sum_{i,j=1}^{\infty} u_{i,j}(\theta)\, \vartheta_{i,j}(\eta,\xi)\Bigg) \vartheta_{n,m}(\eta,\xi) d\eta d\xi d\theta ds \Bigg] +$$

$$+Q_t\, Q_T^{-1} \Bigg[\int\limits_0^T R(s)\, u_{n,m}(s) ds - q_0^{-1} \int\limits_0^T H(s)\, Q_s\, Q_T^{-1} \int\limits_0^T R(\theta)\, u_{n,m}(\theta) d\theta ds \Bigg] \Bigg\}. \tag{2.6}$$

3 Main theorems

We will use the concepts of the following well-known Banach spaces. Space $B_2[0,T]$ of sequences of continuous functions $\{u_{n,m}(t)\}_{n,m=1}^{\infty}$ on the segment $[0,T]$ with norm

$$\| \vec{u}(t) \|_{B_2[0,T]} = \sqrt{\sum_{n,m=1}^{\infty} \left(\max_{t\in[0,T]} | u_{n,m}(t) | \right)^2} < \infty.$$

Hilbert coordinate space ℓ_2 of number sequences $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ with norm

$$\| \vec{\varphi} \|_{\ell_2} = \sqrt{\sum_{n,m=1}^{\infty} | \varphi_{n,m} |^2} < \infty.$$

The space $L_2\left(\Omega_l^2\right)$ of square-summable functions on the domain $\Omega_l^2 = \Omega_l \times \Omega_l$ with norm

$$\| \vartheta(x,y) \|_{L_2\left(\Omega_l^2\right)} = \sqrt{\int_0^l \int_0^l | \vartheta(x,y) |^2 \, dxdy} < \infty.$$

It is not difficult to prove that Bessel's inequality is valid:

$$\| \vec{\varphi} \|_{\ell_2} = \sqrt{\sum_{n,m=1}^{\infty} | \varphi_{n,m} |^2} \leq \| \varphi(x,y) \|_{L_2\left(\Omega_l^2\right)}. \tag{3.1}$$

Theorem 3.1. *Let be fulfilled the smoothness conditions and following conditions:*

1). $\sum\limits_{n,m=1}^{\infty} \left| \psi_{n,m} \right| < \infty$;

2). $\max \left\{ \int\limits_0^T | R(s) | \, ds ; \int\limits_0^T | H(s) | \, ds \right\} < \infty$;

3). $\| f(t,x,y,u \|_{L_2\left(\Omega\times(-\infty,\infty)\right)} \leq C_0 = \text{const}$;

4). $| f(t,x,y,u_1) - f(t,x,y,u_2) | \leq C(x,y) \, |u_1 - u_2 |, \quad 0 < C(x,y) \in L_2\left(\Omega_l^2\right)$;

5). $\rho = \overline{M}_1 \left\| C(x,y) \right\|_{L_2[0,l]^2} \Delta_1 + \Delta_2 < 1$, *where*

$\overline{M}_1 = \sum\limits_{n,m=1}^{\infty} \dfrac{1}{\lambda_{n,m}^2}$, Δ_1 *and* Δ_2 *will be defined below from* (3.6) *and* (3.7)*, respectively.*

Then CSNIE (2.5) *has a unique solution in the space* $B_2[0,T]$.

Proof. In proof of the theorem, we use the method of successive approximations:

$$\begin{cases} u_{n,m}^0(t) = q_0^{-1} \, Q_t \, Q_T^{-1} \, \psi_{n,m}, \\ u_{n,m}^{k+1}(t) = J\left(t; u_{n,m}^k\right), \quad k = 0, \, 1, \, 2, \, 3, \, \ldots, \quad t \in [0,T]. \end{cases} \tag{3.2}$$

Let us estimate the zero approximation $u_{n,m}^0(t)$. We use the known estimate [14]

$$\frac{1}{1+\Gamma(1-\alpha)\tau} \leq E_{\alpha,m,m-1}(-\tau) \leq \frac{1}{1 + \dfrac{\Gamma(1+\alpha(m-1))}{\Gamma(1+\alpha m)}\tau},$$

which is true for every $\alpha \in [0,1]$, $m > 0$ and $\tau \geq 0$. We put $\tau = -\mu_n^2 t^{\alpha+\beta}$, $m = 1 + \dfrac{\beta}{\alpha}$, $\Gamma(1 + \alpha(m-1)) = \Gamma(1+\beta)$, $\Gamma(1+\alpha m) = \Gamma(1+\alpha+\beta)$, then we obtain

$$E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left(-\mu_{n,m}^2 t^{\alpha+\beta} \right) \leq \frac{1}{1 + \dfrac{\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} \mu_{n,m}^2 t^{\alpha+\beta}} \leq 1. \tag{3.3}$$

From (3.3) it is obvious that $0 < \left| Q_t \, Q_T^{-1} \right| < 1$. So, taking into account (1.13), for the first approximation from (3.2) we have

$$\| \vec{u}^0(t) \|_{B_2[0,T]} = \sqrt{\sum_{n,m=1}^{\infty} \left[\max_{0 \leq t \leq T} | \, u_{n,m}^0(t) \, | \right]^2} \leq \sum_{n,m=1}^{\infty} \max_{0 \leq t \leq T} | \, u_{n,m}^0(t) \, | \leq$$

$$\leq \sum_{n,m=1}^{\infty} \max_{0 \leq t \leq T} \left| q_0^{-1} \right| \left| Q_t \, Q_T^{-1} \right| \left| \psi_{n,m} \right| \leq \left| q_0^{-1} \right| \sum_{n,m=1}^{\infty} | \, \psi_{n,m} \, | = \delta_0 < \infty. \tag{3.4}$$

Due to the estimates (1.13), (3.4), applying the Cauchy–Shwartz inequality and Bessel's inequality (3.1), for the first difference we obtain

$$\| \vec{u}^1(t) - \vec{u}^0(t) \|_{B_2[0,T]} \leq$$

$$\leq \sum_{n,m=1}^{\infty} \max_{0 \leq t \leq T} \frac{1}{\lambda_{n,m}^2} \left[\left| \int_0^T \overline{K}(t,s) \int_0^l \int_0^l F\left(t, \eta, \xi, \sum_{i,j=1}^{\infty} u_{i,j}^0(t) \, \vartheta_{i,j}(\eta, \xi) \right) \vartheta_{n,m}(\eta, \xi) d\eta d\xi ds \right| + \right.$$

$$\left. + \left| q_0^{-1} \right| \left| \int_0^T H(s) \int_0^T \overline{K}(s, \theta) \int_0^l \int_0^l F\left(t, \eta, \xi, \sum_{i,j=1}^{\infty} u_{i,j}(t) \, \vartheta_{i,j}(\eta, \xi) \right) \vartheta_{n,m}(\eta, \xi) d\eta d\xi d\theta ds \right| \right] +$$

$$+ \sum_{n,m=1}^{\infty} \max_{0 \leq t \leq T} \left[\left| \int_0^T R(s) \, | \, u_{n,m}^0(s) \, | ds \right| + \left| q_0^{-1} \right| \left| \int_0^T H(s) \, Q_s \, Q_T^{-1} \int_0^T R(\theta) \, | \, u_{n,m}^0(\theta) \, | d\theta ds \right| \right] \leq$$

$$\leq M_1 \max_{0 \leq t \leq T} \left\| \vec{f}(t, \cdot) \right\|_{B_2[0,T]} \left\{ \int_0^T | \, \overline{K}(t,s) \, | ds + \left| q_0^{-1} \right| \int_0^T | \, H(s) \, | \int_0^T | \, \overline{K}(s, \theta) \, | d\theta ds \right\} +$$

$$+ \sum_{n,m=1}^{\infty} \max_{0 \leq t \leq T} | \, u_{n,m}^0(t) \, | \left\{ \int_0^T | \, R(s) \, | ds + \left| q_0^{-1} \right| \int_0^T | \, H(s) \, | \int_0^T | \, R(\theta) \, | d\theta ds \right\} \leq$$

$$\leq \delta_0 \left[M_1 C_0 \Delta_1 + \Delta_2 \right] = \delta_0 \delta_1 < \infty, \tag{3.5}$$

where

$$\delta_1 = M_1 C_0 \Delta_1 + \Delta_2, \quad M_1 = \sqrt{\sum_{n,m=1}^{\infty} \frac{1}{\lambda_{n,m}^4}},$$

$$\Delta_1 = \int_0^T |\, \overline{K}(t,s)\,| ds + \left| q_0^{-1} \right| \int_0^T |\, H(s)\,| \int_0^T |\, \overline{K}(s,\theta)\,| d\theta ds, \tag{3.6}$$

$$\Delta_2 = \int_0^T |\, R(s)\,| ds + \left| q_0^{-1} \right| \int_0^T |\, H(s)\,| \int_0^T |\, R(\theta)\,| d\theta ds. \tag{3.7}$$

Due to the estimate (3.5), for the arbitrary difference $u_{n,m}^{k+1}(t) - u_{n,m}^{k}(t)$ we obtain the following estimate

$$\left\| \vec{u}^{k+1}(t) - \vec{u}^{k}(t) \right\|_{B_2[0,T]} \leq \sum_{n,m=1}^{\infty} \max_{0\leq t\leq T} \left| u_{n,m}^{k+1}(t) - u_{n,m}^{k}(t) \right| \leq$$

$$\leq \sum_{n,m=1}^{\infty} \max_{0\leq t\leq T} \frac{1}{\lambda_{n,m}^2} \Bigg[\Bigg| \int_0^T \overline{K}(t,s) \int_0^l \int_0^l C(\eta,\xi) \sum_{i,j=1}^{\infty} \left| u_{i,j}^{k}(s) - u_{i,j}^{k-1}(s) \right| \vartheta_{i,j}(\eta,\xi)\, \vartheta_{n,m}(\eta,\xi) d\eta d\xi ds \Bigg| +$$

$$+ \left| q_0^{-1} \right| \Bigg| \int_0^T H(s) \int_0^T \overline{K}(s,\theta) \int_0^l \int_0^l C(\eta,\xi) \sum_{i,j=1}^{\infty} \left| u_{i,j}^{k}(\theta) - u_{i,j}^{k-1}(\theta) \right| \vartheta_{i,j}(\eta,\xi)\, \vartheta_{n,m}(\eta,\xi) d\eta d\xi d\theta ds \Bigg| \Bigg] +$$

$$+ \sum_{n,m=1}^{\infty} \max_{0\leq t\leq T} \Bigg[\Bigg| \int_0^T |\, R(s)\,| \left| u_{i,j}^{k}(s) - u_{i,j}^{k-1}(s) \right| ds \Bigg| + \left| q_0^{-1} \right| \Bigg| \int_0^T H(s) \int_0^T |\, R(\theta)\,| \left| u_{i,j}^{k}(\theta) - u_{i,j}^{k-1}(\theta) \right| d\theta ds \Bigg| \Bigg] \leq$$

$$\leq \frac{2}{l} \overline{M}_1 \max_{0\leq t\leq T} \left\| C(x,y) \right\|_{B_2[0,T]} \left\| \vec{u}^{k}(t) - \vec{u}^{k-1}(t) \right\|_{B_2[0,T]} \Delta_1 + \sum_{n,m=1}^{\infty} \max_{0\leq t\leq T} \left| u_{n,m}^{k}(t) - u_{n,m}^{k-1}(t) \right| \Delta_2 \leq$$

$$\leq \delta_0 \delta_1 \left[\frac{2}{l} \overline{M}_1 \left\| C(x,y) \right\|_{L_2[0,l]^2} \Delta_1 + \Delta_2 \right]^k = \delta_0 \delta_1 \rho^k, \tag{3.8}$$

where

$$\rho = \frac{2}{l} \overline{M}_1 \left\| C(x,y) \right\|_{L_2[0,l]^2} \Delta_1 + \Delta_2, \quad \overline{M}_1 = \sum_{n,m=1}^{\infty} \frac{1}{\lambda_{n,m}^2}.$$

Since $\rho = \frac{2}{l} \overline{M}_1 \left\| C(x,y) \right\|_{L_2[0,l]^2} \Delta_1 + \Delta_2 < 1$ and $\delta_0 < \infty$, $\delta_1 < \infty$, from estimates (3.8) follows that

$$\lim_{k\to\infty} \left\| \vec{u}^{k+1}(t) - \vec{u}^{k}(t) \right\|_{B_2[0,T]} = 0. \tag{3.9}$$

From estimates (3.4), (3.5) and (3.9) we obtain the existence and uniqueness of the solution $\vec{u}(t) \in B_2[0,T]$ to CSNIE (2.5). Theorem 3.1 is proved. □

The Fourier series (2.6) we consider as a formal solution of the problem (0.1)–(0.3).

Theorem 3.2. *Let the conditions of Theorem 3.1 be satisfied and*

$$2\delta_0 \left[1 + \delta_1 \left(\frac{2}{l} \overline{M}_1 \left\| C(x,y) \right\|_{L_2[0,l]^2} \Delta_1 + \Delta_2 \right) \right] < l.$$

If $\vec{u}(t) \in B_2[0,T]$ is the unique solution to CSNIE (2.5), *then the series* (2.6) *will be generalized solution to the mixed problem* (0.1)–(0.3) *almost everywhere.*

Proof. We consider the difference

$$\Big| U(t,x,y) - U^k(t,x,y) \Big| = \Big| \sum_{n,m=1}^{\infty} u_{n,m}^{\infty}(t)\,\vartheta_{n,m}(x,y) - \sum_{n,m=k}^{\infty} u_{n,m}^{k}(t)\,\vartheta_{n,m}(x,y) \Big| \le$$

$$\le \sum_{n,m=k}^{\infty} \Big| u_{n,m}(t)\,\vartheta_{n,m}(x,y) \Big|.$$

As in the case of proof of the theorem 3.1, we obtain the estimate

$$\sum_{n,m=k}^{\infty} \Big| u_{n,m}(t)\,\vartheta_{n,m}(x,y) \Big| \le \sum_{n,m=k}^{\infty} |\,\vartheta_{n,m}(x,y)\,| \times$$

$$\times \left\{ \Big| q_0^{-1} \Big| \sum_{n,m=k}^{\infty} |\,\psi_{n,m}\,| + \delta_0 \delta_1 \Big[\frac{2}{l} \overline{M}_1 \Big\| C(x,y) \Big\|_{L_2[0,l]^2} \Delta_1 + \Delta_2 \Big] \right\} \le$$

$$\le \frac{2}{l} \delta_0 \Big[1 + \delta_1 \Big(\frac{2}{l} \overline{M}_1 \Big\| C(x,y) \Big\|_{L_2[0,l]^2} \Delta_1 + \Delta_2 \Big) \Big] < \infty. \tag{3.10}$$

Since $\vec{u}(t) \in B_2[0,T]$ and $\frac{2}{l} \delta_0 \Big[1 + \delta_1 \Big(\frac{2}{l} \overline{M}_1 \Big\| C(x,y) \Big\|_{L_2[0,l]^2} \Delta_1 + \Delta_2 \Big) \Big] < 1$ from (3.10) implies that

$$\lim_{k\to\infty} \Big| U(t,x,y) - U^k(t,x,y) \Big| = 0.$$

This proved the theorem 3.2. □

Now the function $u_{n,m}(t)$ in the representation (2.4) is known. As $\vec{u}(t) \in B_2[0,T]$, the proof of the convergence of series (2.4) is similarly tho the proof of the Theorem 3.2.

Conclusion

In the domain $\Omega = \big\{(t,x,y)\,|\,0 < t < T,\ 0 < x,y < l\big\}$ the generalized solvability of the inverse mixed problem (0.1)–(0.4) for a partial differential equation is considered for the case of α-order Gerasimov–Caputo type fractional operator with order $0 < \alpha \le 1$. The considering equations depend from three independent arguments. First argument is time argument and with respect to this argument considering equation is fractional Gerasimov–Caputo type ordinary differential equation. Second and third arguments are spatial and the equations with respect to these arguments are differential equations of second order. The Fourier series method is used and a countable system of differential equations is obtained and studied. The generalized solvability of mixed problem (0.1)–(0.3) is derived in the form of the Fourier series.

Declarations

Funding: This research has been funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23488740, 2024-2026 years).

Data availability: This manuscript has no associated data.

Ethical Conduct: Not applicable.

Conflicts of interest: The authors declare that there is no conflict of interest.

References

1. V. M. Aleksandrov, E. V. Kovalenko, *Problems of continuum mechanics with mixed boundary conditions*. Nauka, Moscow, 1986. [in Russian].

2. V. A. Ilyin, "On the solvability of mixed problems for hyperbolic and parabolic equations" [in Russian], *Uspekhi Matematicheskikh Nauk* **15** (2), 97–154 (1960).

3. V. A. Chernyatin, *Justification of the Fourier method in a mixed problem for partial differential equations*. MSU, Moscow, 1992. [in Russian]

4. A. I. Vagabov, "Generalized Fourier method for solving mixed problems for nonlinear equations", *Differential Equations* **32** (1), 94–105 (1996).

5. T. K. Yuldashev, "Mixed value problem for nonlinear integro-differential equation with parabolic operator of higher power", *Computational Mathematics and Mathematical Physics* **52** (1), 105–116 (2012).

6. G. I. Barenblatt, Yu. P. Zheltov, I. N. Kochina, "On finitness conditions in the mechanics of continuous media. Static problems of the theory of elasticity", *Prikladnaya Matematika i Mekhanika* **24** (5), 316–322 (1960). [in Russian]

7. G. I. Barenblatt, Yu. P. Zheltov, "Fundamental equations of filtration of homogeneous liquids in fissured rocks", *Soviet Physics, Doklady* **132** (5), 522–525 (1960).

8. E. N. Bereslavskii, "Effect of evaporation or infiltration on the free surface of groundwater in certain problems of underground hydromechanics", *American Journal of Applied Mathematics and Statistics* **5**, (5), 159–163 (2017). https://doi.org/10.12691/ajams-5-5-1

9. G. I. Barenblatt, V. M. Yentov, V. M. Ryzhik, *Movement of liquids and gases in natural reservoirs*. Nedra, Moscow, 1984. [in Russian]

10. K. S. Basniev, I. N. Kochina, V. M. Maksimov, *Underground hydromechanics*. Nedra, Moscow, 1993. [in Russian]

11. P. Ya. Polubarinova–Kochina, *The theory of groundwater movement*. Nauka, Moscow, 1977. [in Russian]

12. G. A. Sviridyuk, D. E. Shafranov, "The Cauchy problem for the Barenblatt–Zheltov–Kochina equation on a smooth manifold", *Vestnik Chelyabinskogo Gosudarstvennogo Universiteta* **9**, 171–177 (2003). [in Russian]

13. M. A. Sagadeeva, F. L. Hasan, "Bounded solutions of Barenblatt–Zheltov–Kochina model in Quasi–Sobolev spaces", *Bulletin of the South Ural State University. Seria Mathematical Modelling, Programming and Computer Software* **8** (4), 138–144 (2015).

14. R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogozin, *Mittag–Leffler functions, Related Topics and Applications*, Springer, Berlin – Heidelberg, Germany (2014) https://doi.org/10.1007/978-3-662-61550-8.

15. *Handbook of fractional calculus with applications in 8 volumes* (ed. by J. A. Tenreiro Machado), Walter de Gruyter GmbH, Berlin – Boston, 2019, 47–85.

16. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*. Elsevier, North-Holland, Mathematics studies, 2006.

17. D. Kumar, D. Baleanu, "Fractional calculus and its applications in physics", *Frontier physics* **7** (6) (2019). https://doi.org/10.3389/fphy.2019.00081.

18. C. Lizama, *Abstract linear fractional evolution equations. Handbook of fractional calculus with applications* J. A. T. Marchado Ed., DeGruyter, Berlin, **2**, 465–497 (2019).

19. Myong-Ha Kim, Guk-Chol Ri, O. Hyong-Chol, "Operational method for solving multi-term fractional differential equations with the generalized fractional derivatives", *Fractional Calculis and its Applicable Analysis* **17** (1), 79–95 (2014). https://doi.org/10.2478/s13540-014-0156-6.

20. S. Patnaik, J. P. Hollkamp, F. Semperlotti, "Applications of variable-order fractional operators: a review", *Proceedings A, Royal Society* **A476** (20190498), 1–24 (2020). http://dx.doi.org/10.1098/rspa.2019.0498

21. R. K. Saxena, R. Garra, E. Orsingher, "Analytical solution of space-time fractional telegraph-type equations involving Hilfer and Hadamard derivatives", *Integral transforms and special functions* No. 6 (2015). https://doi.org/10.1080/10652469.2015.1092142.

22. H. Sun, A. Chang, Y. Zhang, W. Chen, "A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications", *Fractional Calculis and its Applicable Analysis* **22**, 27–59 (2019). https://doi.org/10.1515/fca-2019-0003

23. O. Kh. Abdullaev, O. Sh. Salmanov, T. K. Yuldashev, "Direct and inverse problems for a parabolic-hyperbolic equation involving Riemann–Liouville derivatives", *Transactions of National Academy of Sciens of Azerbaijan. Seria Physical-Technical and Mathematical Science. – Mathematics* **43** (1), 21–33 (2023).

24. O. Kh. Abdullaev, T. K. Yuldashev, "Inverse problems for the loaded parabolic-hyperbolic equation involves Riemann–Liouville operator", *Lobachevskii Journal of Mathematics* **44** (3), 1080–1090 (2023). https://doi.org/10.1134/S1995080223030034

25. B. Ahmad, A. Alsaedi, M. Kirane, R. G. Tapdigoglu, "An inverse problem for space and time fractional evolution equations with an involution perturbation", *Quaestiones Mathematicae* **40** (2), 151–160 (2017). https://doi.org/10.2989/16073606.2017.1283370

26. A. S. Berdyshev, J. B. Kadirkulov, "On a nonlocal problem for a fourth-order parabolic equation with the fractional Dzhrbashyan–Nersesyan operator", *Differential Equations* **52** (1), 122–127 (2016). https://doi.org/10.1134/S0012266116010109

27. S. A. Berdyshev, A. Cabada, J. B. Kadirkulov, "The Samarskii–Ionkin type problem for the fourth order parabolic equation with fractional differential operator", *Computers and Mathematics with Applications* **62**, 3884–3893 (2011).

28. M. Kh. Beshtokov, "To boundary-value problems for degenerating pseudoparabolic equations with Gerasimov-Caputo fractional derivative", *Russian Mathematics (Izv. vuz)* **62** (10), 1–14, (2018). https://doi.org/10.3103/S1066369X18100018

29. P. N. Duc, H. D. Binh, L. D. Long, H. T. Kim Van, "Reconstructing the right-hand side of the Rayleigh–Stokes problem with nonlocal in time condition", *Advances in Difference Equations* **470**, 1–18 (2021). https://doi.org/10.1186/s13662-021-03626-z

30. V. E. Fedorov, A. V. Nagumanova, "Inverse linear problems for a certain class of degenerate fractional evolution equations", *Itogi nauki i tekhniki. Seriya Sovremennaya matematika i ikh prilozheniya. Tematicheskiye obzory* **167**, 97–111 (2019). [in Russian] https://doi.org/10.36535/0233-6723-2019-167-97-111

31. S. Kerbal, B. J. Kadirkulov, M. Kirane, "Direct and inverse problems for a Samarskii–Ionkin type problem for a two dimensional fractional parabolic equation", *Progress in Fractional Differentiation and Applications* **4** (3), 1–14 (2018). http://dx.doi.org/10.18576/pfda/01010

32. Y. Liu, Zh. Li, M. Yamamoto, *Inverse Problems of Determining Sources of the Fractional Partial Differential Equations. Handbook of Fractional Calculus with Applications* J. A. T. Marchado Ed. DeGruyter, Berlin (2019), **2**, 411–429 (2019).

33. M. Kosmakova, D. Akhmanova, K. Izhanova, "BVP with a load in the form of a fractional integral" *International Journal of Mathematics and Mathematical Sciences* **2024** (7034103), (2024).

34. N. K. Ochilova, T. K. Yuldashev, "On a nonlocal boundary value problem for a degenerate parabolic-hyperbolic equation with fractional derivative", *Lobachevskii Journal of Mathematics* **43** (1), 229–236 (2022). https://doi.org/10.1134/S1995080222040175

35. N. T. Orumbayeva, M. T. Kosmakova, T. D. Tokmagambetova, A. M. Manat, "Solutions of boundary value problems for loaded hyperbolic type equations", *Bulletin of the Karaganda University. Mathematics* **118** (2), 177–188 (2025).

36. A. V. Pskhu, *Fractional partial differential equations.* Nauka, Moscow, 2005. [in Russian]

37. M. Ruzhansky, N. Tokmagambetov, B. T. Torebek, "Inverse source problems for positive operators. I: Hypoelliptic diffusion and subdiffusion equations", *Journal of Inverse Ill-Possed Problems* **27**, 891–911 (2019).

38. T. K. Yuldashev, O. Kh. Abdullaev, "Unique solvability of a boundary value problem for a loaded fractional parabolic-hyperbolic equation with nonlinear terms", *Lobachevskii Journal of Mathematics* **42** (5), 1113–1123 (2021). https://doi.org/10.1134/S1995080221050218

39. T. K. Yuldashev, T. A. Abduvahobov, "Periodic solutions for an impulsive system of fractional order integro-differential equations with maxima", *Lobachevskii Journal of Mathematics* **44** (10), 4401–4409 (2023). https://doi.org/10.1134/S1995080223100451

40. T. K. Yuldashev, B. J. Kadirkulov, "On a boundary value problem for a mixed type fractional differential equations with parameters", *Proceedings of the Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan* **47** (1), 112–123 (2021).

41. T. K. Yuldashev, B. J. Kadirkulov, R. A. Bandaliyev, "On a mixed problem for Hilfer type fractional differential equation with degeneration", *Lobachevskii Journal of Mathematics* **43** (1), 263–274 (2022). https://doi.org/10.1134/S1995080222040229

42. T. K. Yuldashev, T. G. Ergashev, T. A. Abduvahobov, "Nonlinear system of impulsive integro-differential equations with Hilfer fractional operator and mixed maxima", *Chelyabinsk Physycal and Mathematical Journal* **7** (3), 312–325 (2022).

43. Y. Zhang, X. Xu, "Inverse source problem for a fractional differential equations", *Inverse Probems* **27** (3), 31–42 (2011).

44. *Theory and applications of fractional differential equations.* Hilfer R (ed.). World Scientific Publishibg Company, Singapore, 2000. https://doi.org/10.1142/3779.

45. L. Boudabsa, T. Simon "Some properties of the Kilbas–Saigo function", *Mathematics* **9** (217), 1–24 (2021).